On Some Hypergeometric Summations II. Duality and Reciprocity*

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Abstract

We continue our studies on non-terminating hypergeometric summations with one free parameter, where the concepts of duality and reciprocity play pivotal roles in this article. They are used not only to extend the previously obtained results to a larger region but also to strengthen the results substantially, regarding arithmetic constraints for a given hypergeometric sum to admit a gamma product formula (GPF). Among other things we are able to settle the rationality and finiteness conjectures posed in the previous paper.

1 Introduction

Inspired by a series of works by Ebisu [3, 4, 5], Iwasaki [8] set out to develop a theoretical study of non-terminating hypergeometric summations with one free parameter. In the present article we continue this study and settle some of the open problems posed there, where two new symmetries which we call *duality* and *reciprocity* will play central roles.

The origin of these symmetries is very simple. The Gauss hypergeometric equation

$$z(1-z)\frac{du}{dz^2} + \{\gamma - (\alpha+\beta+1)z\}\frac{du}{dz} - \alpha\beta\, u = 0 \tag{1}$$

has the Riemann scheme:

$$\begin{Bmatrix} z=0 & z=1 & z=\infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{Bmatrix}, \tag{2}$$

in which the hypergeometric series ${}_2F_1(\alpha,\beta;\gamma;z)$ is just the solution of local exponent 0 at the origin $z=0$. Now the idea is this: exchanging the solution of exponent 0 at $z=0$ with that of exponent $1-\gamma$ at the same point $z=0$ produces duality, while exchanging the solution of exponent 0 at $z=0$ with that of exponent $\gamma-\alpha-\beta$ at $z=1$ gives rise to reciprocity.

The purpose of this article is to explain how these two symmetries can be used not only to extend our previous results to a larger domain, but also to strengthen themselves substantially

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in the original domain. In particular we are able to show that they are powerful enough to settle the *rationality* and *finite-cardinality* conjectures for the numbers a and b in our notation below (see Theorem 3.10 among other results).

We recall the main issues of our study. For a data $\lambda = (p, q, r; a, b, c; x) \in \mathbb{R}^6 \times (-1, 1)$, let

$$f(w; \lambda) := {}_2F_1(pw + a,\ qw + b;\ rw + c;\ x).$$

Putting the case $r = 0$ aside, we assume $r \neq 0$. Then we may assume $r > 0$, for otherwise a simultaneous change of signs $(p, q, r; w) \mapsto (-p, -q, -r; -w)$ makes $r > 0$. Moreover we may assume $c = 0$ by a shift of variable $w \mapsto w - c/r$. Thus from the outset we have only to consider

$$f(w; \lambda) := {}_2F_1(pw + a,\ qw + b;\ rw;\ x), \tag{3}$$

where the revised data $\lambda = (p, q, r; a, b; x)$ ranges over the domain

$$p,\ q \in \mathbb{R}; \qquad r > 0; \qquad a,\ b \in \mathbb{R}; \qquad -1 < x < 0 \quad \text{or} \quad 0 < x < 1, \tag{4}$$

with the trivial case $x = 0$ being excluded. We are interested in the following problems.

Problem I Find a data $\lambda = (p, q, r; a, b; x)$ for which $f(w; \lambda)$ has a *gamma product formula*:

$$f(w; \lambda) = C \cdot d^w \cdot \frac{\Gamma(w + u_1) \cdots \Gamma(w + u_m)}{\Gamma(w + v_1) \cdots \Gamma(w + v_n)}, \tag{5}$$

for some $C,\ d \in \mathbb{C}^\times$; $m,\ n \in \mathbb{Z}_{\geq 0}$; $u_1, \dots, u_m \in \mathbb{C}$; and $v_1, \dots, v_n \in \mathbb{C}$.

Problem I is something like the peak of a high mountain, which is difficult to climb up directly. We need a base camp for attacking it and this role is played by the following.

Problem II Find a data $\lambda = (p, q, r; a, b; x)$ for which $f(w; \lambda)$ is *of closed form*, that is,

$$\frac{f(w + 1; \lambda)}{f(w; \lambda)} =: R(w; \lambda) \in \mathbb{C}(w) \ \ : \ \text{a rational function of } w. \tag{6}$$

It is evident from recursion formula $\Gamma(w + 1) = w\,\Gamma(w)$ that any solution to Problem I leads to a solution to Problem II, where the corresponding rational function is given by

$$R(w; \lambda) = d \cdot \frac{(w + u_1) \cdots (w + u_m)}{(w + v_1) \cdots (w + v_n)}. \tag{7}$$

However the converse question is by no means trivial. We say that a solution (6) to Problem II with rational function (7) leads back to a solution (5) to Problem I, if there exists a constant $C \in \mathbb{C}^\times$ such that formula (5) holds true with d; $u_1, \dots, u_m$; and $v_1, \dots, v_n$ coming from (7).

Problem 1.1 When does a solution to Problem II lead back to a solution to Problem I ?

A solution λ to Problem I or II is said to be *elementary*, if the function $f(w; \lambda)$ has at most finitely many poles in $\mathbb{C}_w$; otherwise, it is said to be *non-elementary*.

Problem 1.2 Enumerate all elementary solutions and determine them explicitly.

For a non-elementary solution the existence of infinitely many poles allows us to discuss asymptotics and dynamics of their residues as the poles tends to infinity. This brings us significant information about the solution under consideration (see Iwasaki [8, §7–§10]).

Definition 1.3 A data $\lambda = (p, q, r; a, b; x)$ is said to be *integral* if $\boldsymbol{p} := (p, q; r) \in \mathbb{Z}^3$; *rational* if $\boldsymbol{p} \in \mathbb{Q}^3$; and *irrational* if $\boldsymbol{p} \notin \mathbb{Q}^3$. One can speak of an integral, rational or irrational solution to Problem I or II. Note that the component $(a, b; x)$ has nothing to do with this definition.

There is an efficient method to find *integral* solutions to Problem II. To describe it we simplify the notation by writing ${}_2F_1(\boldsymbol{a}; z) = {}_2F_1(\alpha, \beta; \gamma; z)$ with $\boldsymbol{a} = (a_1, a_2; a_3) = (\alpha, \beta; \gamma)$. A *three-term relation* for ${}_2F_1(\boldsymbol{a}; z)$ is an identity of the form:

$$ {}_2F_1(\boldsymbol{a} + \boldsymbol{p}; z) = r(\boldsymbol{a}; z)\, {}_2F_1(\boldsymbol{a}; z) + q(\boldsymbol{a}; z)\, {}_2F_1(\boldsymbol{a} + \boldsymbol{1}; z), \tag{8} $$

where $\boldsymbol{p} = (p, q; r) \in \mathbb{Z}^3$ is an integer vector; $\boldsymbol{1} := (1, 1; 1)$; and $q(\boldsymbol{a}; z)$ and $r(\boldsymbol{a}; z)$ are rational functions of $(\boldsymbol{a}; z)$ uniquely determined by $\boldsymbol{p}$. Relation (8) can be obtained by composing a finite sequence of fifteen contiguous relations of Gauss (see Andrews et al. [1, §2.5]). Ebisu [3] found an efficient formula for the coefficients $q(\boldsymbol{a}; z)$ and $r(\boldsymbol{a}; z)$ (see also Vidunas [12]).

For an integral data $\lambda = (p, q, r; a, b; x)$, substituting $\boldsymbol{a} = \boldsymbol{\alpha}(w) := (pw + a, qw + b; rw)$ and $z = x$ into equation (8) we have

$$ f(w + 1; \lambda) = R(w; \lambda)\, f(w; \lambda) + Q(w; \lambda)\, \tilde{f}(w; \lambda), \tag{9} $$

with $\tilde{f}(w; \lambda) := {}_2F_1(\boldsymbol{\alpha}(w) + \boldsymbol{1}; x)$, where $Q(w; \lambda) := q(\boldsymbol{\alpha}(w); x)$ and $R(w; \lambda) := r(\boldsymbol{\alpha}(w); x)$ are rational functions of w depending uniquely on λ. If λ happens to be such a data that

$$ Q(w; \lambda) = 0 \qquad \text{in} \quad \mathbb{C}(w), \tag{10} $$

then three-term relation (9) reduces to a two-term one (6), yielding a solution to Problem II. An integral solution to Problem II so obtained is said to *come from contiguous relations*. Finding a solution in this manner is called the *method of contiguous relations*. Condition (10) yields an overdetermined system of algebraic equations for $(a, b; x)$ with a given triple $(p, q, r) \in \mathbb{Z}^3$.

Problem 1.4 When does an integral solution to Problem II come from contiguous relations ?

Given a data $\lambda = (p, q, r; a, b; x)$ and a positive integer $k \in \mathbb{N}$, the new data $k\lambda := (kp, kq, kr; a, b; x)$ is referred to as the *multiplication* of λ by k. It is said to be *nontrivial* if $k \geq 2$, in particular 2λ is the *duplication* of λ. In view of definition (3) we have

$$ f(w; k\lambda) = f(kw; \lambda) \qquad (k \in \mathbb{N}). \tag{11} $$

Gauss's multiplication formula for the gamma function [1, Theorem 1.5.2]:

$$ \Gamma(kw) = (2\pi)^{(1-k)/2} \cdot k^{kw-1/2} \cdot \prod_{j=0}^{k-1} \Gamma\left(w + \tfrac{j}{k}\right) \qquad (k \in \mathbb{N}) \tag{12} $$

implies that if λ is a solution to Problem I with gamma product formula (5) then $k\lambda$ is also a solution to the same problem with the "multiplied" gamma product formula

$$ f(w; k\lambda) = C_k \cdot d_k^w \cdot \prod_{j=0}^{k-1} \frac{\Gamma\left(w + \frac{u_1+j}{k}\right) \cdots \Gamma\left(w + \frac{u_m+j}{k}\right)}{\Gamma\left(w + \frac{v_1+j}{k}\right) \cdots \Gamma\left(w + \frac{v_n+j}{k}\right)}, \tag{13} $$

where $C_k := C \cdot (2\pi)^{(k-1)(n-m)/2} \cdot k^{u-v+(n-m)/2}$ with $u := u_1 + \cdots + u_m$, $v := v_1 + \cdots + v_n$, and $d_k := d^k \cdot k^{k(m-n)}$. In a similar manner, if λ is a solution to Problem II with rational function $R(w; \lambda)$ as in condition (6) then $k\lambda$ is also a solution to the same problem with

$$R(w; k\lambda) = \prod_{j=0}^{k-1} R(kw + j; \lambda) = d_k \cdot \prod_{j=0}^{k-1} \frac{\left(w + \frac{u_1+j}{k}\right) \cdots \left(w + \frac{u_m+j}{k}\right)}{\left(w + \frac{v_1+j}{k}\right) \cdots \left(w + \frac{v_n+j}{k}\right)}. \tag{14}$$

For any rational solution $\lambda = (p, q, r; a, b; x)$ to Problem II there exists a positive integer k such that $k\boldsymbol{p} := (kp, kq; kr) \in \mathbb{Z}^3$, so it makes sense to ask whether the integral solution $k\lambda$ comes from contiguous relations. If the answer is "yes" for some $k \in \mathbb{N}$, we say that the rational solution λ *essentially comes from contiguous relations*. As a variant of Problem 1.4 we pose:

Problem 1.5 When does a rational solution to Problem II *essentially* come from contiguous relations ?

It is easy to see that formula (5) can be recovered from formula (13) via relation (11). Indeed, we have only to replace w by w/k in (13) and use the multiplication formula (12) in the other way round (see Lemma 10.5 for a relevant discussion). However, it is totally unclear whether formula (7) can be recovered from formula (14), because with respect to the multiplication $R(w; \lambda)$ has no such simple structure as the relation (11) for $f(w; \lambda)$, so we have to establish a bilateral link between Problems I and II. This is one of several reasons for the importance of discussing Problem 1.1. We refer to Remark 4.2 for another reason.

2 Symmetries

There are some transformations indispensable in discussing Problems I and II. They consist of well-known classical symmetries in §2.1, as well as new ones we shall introduce in §2.3.

2.1 Classical Symmetries

The hypergeometric series ${}_2F_1(\alpha, \beta; \gamma; z)$ admits the following well-known symmetries:

$$\begin{aligned} {}_2F_1(\alpha,\, \beta;\, \gamma;\, z) &= {}_2F_1(\beta,\, \alpha;\, \gamma;\, z) && \text{(trivial)} && (15\text{a}) \\ &= (1-z)^{\gamma-\alpha-\beta}\, {}_2F_1(\gamma-\alpha,\, \gamma-\beta;\, \gamma;\, z) && \text{(Euler)} && (15\text{b}) \\ &= (1-z)^{-\alpha}\, {}_2F_1(\alpha,\, \gamma-\beta;\, \gamma;\, z/(z-1)) && \text{(Pfaff)} && (15\text{c}) \\ &= (1-z)^{-\beta}\, {}_2F_1(\gamma-\alpha,\, \beta;\, \gamma;\, z/(z-1)). && \text{(Pfaff)} && (15\text{d}) \end{aligned}$$

Compatible with Problems I and II, they induce symmetries on solutions to those problems:

$$\begin{aligned} \lambda = (p, q, r; a, b; x) &\mapsto (q, p, r; b, a; x) && \text{(trivial)} && (16\text{a}) \\ &\mapsto (r-p,\, r-q,\, r;\, -a,\, -b;\, x) && \text{(Euler)} && (16\text{b}) \\ &\mapsto (p,\, r-q,\, r;\, a,\, -b;\, x/(x-1)) && \text{(Pfaff)} && (16\text{c}) \\ &\mapsto (r-p,\, q,\, r;\, -a,\, b;\, x/(x-1)). && \text{(Pfaff)} && (16\text{d}) \end{aligned}$$

The symmetries (16) were already introduced and discussed in Iwasaki [8, §1].

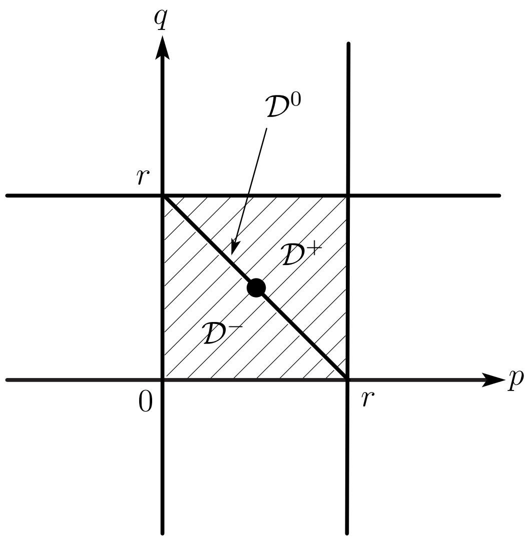

Figure 1: The domain $\mathcal{D}$.

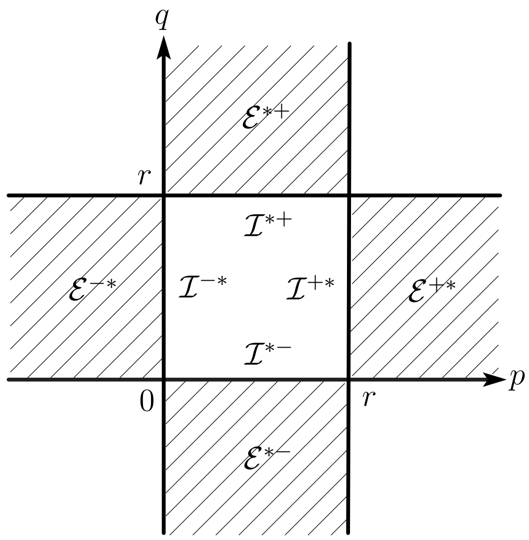

Figure 2: The domain $\mathcal{E}$ and the region $\mathcal{I}$.

2.2 Division into Several Components

As to the argument x in domain (4), Pfaff's transformation (16c) or (16d) takes the negative unit interval $-1 < x < 0$ to a positive one $0 < x < 1/2$, so we have only to work on the positive unit interval $0 < x < 1$. Thus domain (4) can be reduced to the following.

$$p,\, q \in \mathbb{R}; \qquad r > 0; \qquad a,\, b \in \mathbb{R}; \qquad 0 < x < 1. \tag{17}$$

According to the location of (p, q) relative to r the domain (17) is divided into several components, the figures of which are depicted upon projected to the (p, q) plane with a fixed value of $r > 0$ as in Figures 1 and 2. First we consider the "square" domain

$$\mathcal{D} := \{\, \lambda \in \mathbb{R}^6 \,:\, 0 < p < r, \quad 0 < q < r; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\},$$

which is further decomposed into three components $\mathcal{D} = \mathcal{D}^- \cup \mathcal{D}^0 \cup \mathcal{D}^+$, where

$$\begin{aligned}
\mathcal{D}^- &:= \{\, \lambda \in \mathbb{R}^6 \,:\, p > 0, \quad q > 0, \quad p + q < r; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\}, \\
\mathcal{D}^+ &:= \{\, \lambda \in \mathbb{R}^6 \,:\, p < r, \quad q < r, \quad p + q > r; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\}, \\
\mathcal{D}^0 &:= \{\, \lambda \in \mathbb{R}^6 \,:\, p > 0, \quad q > 0, \quad p + q = r; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\}.
\end{aligned}$$

Euler's transformation (16b) swaps $\mathcal{D}^-$ and $\mathcal{D}^+$ while preserving the diagonal $\mathcal{D}^0$. Thus we have only to deal with $\mathcal{D}^- \cup \mathcal{D}^0$. Next we define $\mathcal{E} := \mathcal{E}^{*-} \cup \mathcal{E}^{*+} \cup \mathcal{E}^{-*} \cup \mathcal{E}^{+*}$ by

$$\begin{aligned}
\mathcal{E}^{*-} &:= \{\, \lambda \in \mathbb{R}^6 \,:\, 0 < p < r, \quad q < 0; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\}, \\
\mathcal{E}^{*+} &:= \{\, \lambda \in \mathbb{R}^6 \,:\, 0 < p < r, \quad q > r; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\}, \\
\mathcal{E}^{-*} &:= \{\, \lambda \in \mathbb{R}^6 \,:\, p < 0, \quad 0 < q < r; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\}, \\
\mathcal{E}^{+*} &:= \{\, \lambda \in \mathbb{R}^6 \,:\, p > r, \quad 0 < q < r; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\},
\end{aligned}$$

where the first superscript $-$, $*$ or $+$ designates the interval $p < 0$, $0 < p < r$ or $r < p$ respectively, while the second superscript is subject to the same convention with respect to q.

In a similar manner we define $\mathcal{I} := \mathcal{I}^{*-} \cup \mathcal{I}^{*+} \cup \mathcal{I}^{-*} \cup \mathcal{I}^{+*}$ by

$$\begin{aligned}
\mathcal{I}^{*-} &:= \{\, \lambda \in \mathbb{R}^6 \,:\, 0 < p < r, \quad q = 0; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\}, \\
\mathcal{I}^{*+} &:= \{\, \lambda \in \mathbb{R}^6 \,:\, 0 < p < r, \quad q = r; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\}, \\
\mathcal{I}^{-*} &:= \{\, \lambda \in \mathbb{R}^6 \,:\, p = 0, \quad 0 < q < r; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\}, \\
\mathcal{I}^{+*} &:= \{\, \lambda \in \mathbb{R}^6 \,:\, p = r, \quad 0 < q < r; \quad a,\, b \in \mathbb{R}; \quad 0 < x < 1 \,\}.
\end{aligned}$$

The trivial and Euler's transformations (16a) and (16b) induce the transpositions:

$$\mathcal{E}^{*-} \leftrightarrow \mathcal{E}^{-*}, \quad \mathcal{E}^{*+} \leftrightarrow \mathcal{E}^{+*} \quad \text{by (16a)}; \qquad \mathcal{E}^{*-} \leftrightarrow \mathcal{E}^{*+}, \quad \mathcal{E}^{-*} \leftrightarrow \mathcal{E}^{+*} \quad \text{by (16b)},$$

with the same rules applying to the regions $\mathcal{I}^{\star\star}$. Thus we have only to deal with $\mathcal{E}^{*-} \cup \mathcal{I}^{*-}$.

2.3 Duality and Reciprocity

We introduce two symmetries or transformations, which we call duality and reciprocity.

Definition 2.1 Define a transformation $\lambda = (p, q, r; a, b; x) \mapsto \lambda' := (p', q', r'; a', b'; x')$ by

$$p' := p, \quad q' := q, \quad r' := r; \quad a' := 1 - \frac{2p}{r} - a, \quad b' := 1 - \frac{2q}{r} - b; \quad x' := x, \tag{18}$$

which is referred to as *duality*, where the $(p, q, r; x)$-component of λ is kept unchanged. It is a well-defined involution whenever r does not vanish.

Note that duality maps each of the components $\mathcal{D}^{\pm}$, $\mathcal{D}^0$, $\mathcal{E}^{*\pm}$ and $\mathcal{E}^{\pm*}$ bijectively onto itself.

Given a dual pair (λ, λ'), it is convenient to think of the three-by-two matrix

$$\begin{pmatrix} p & q \\ a & b \\ a' & b' \end{pmatrix}.$$

Exchange of its columns represents the trivial symmetry (16a) applied to (λ, λ'), whereas the exchange of its middle and bottom rows represents duality (18) itself. By the omission of its top row (p, q), the matrix above is abbreviated to the square matrix

$$\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}. \tag{19}$$

Two dual pairs are said to be *equivalent* if their matrices are transitive via the column or row exchange or the composition of them. This concept will be used in §7 to reduce the work there.

Definition 2.2 Define a transformation $\lambda = (p, q, r; a, b; x) \mapsto \check{\lambda} := (\check{p}, \check{q}, \check{r}; \check{a}, \check{b}; \check{x})$ by

$$\begin{gathered}
\check{p} := -p, \qquad \check{q} := -q, \qquad \check{r} := r - p - q; \qquad \check{x} := 1 - x, \\
\check{a} := \frac{(r - q)(1 - a) - pb}{r - p - q}, \qquad \check{b} := \frac{(r - p)(1 - b) - qa}{r - p - q},
\end{gathered} \tag{20}$$

which is referred to as *reciprocity*. This is a well-defined involution on the domain

$$p,\, q,\, r \in \mathbb{R}, \quad r(r - p - q) \neq 0; \qquad a,\, b \in \mathbb{R}; \qquad 0 < x < 1. \tag{21}$$

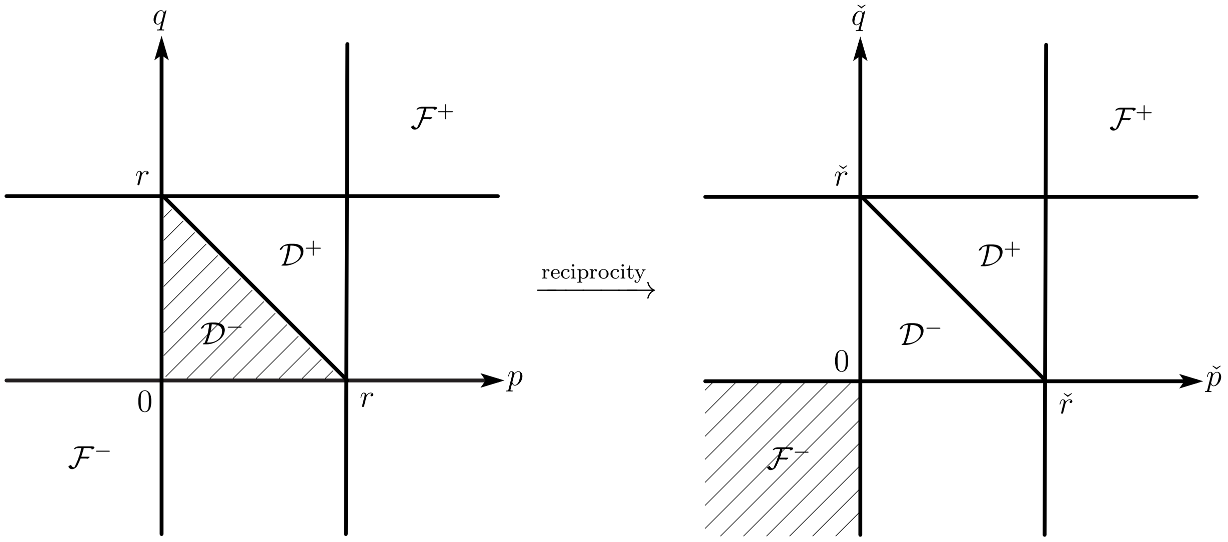

Figure 3: Reciprocity between $\mathcal{D}^-$ and $\mathcal{F}^-$.

As indicated in Figure 3, two new components $\mathcal{F}^{\pm}$ are introduced by

$$\mathcal{F}^- := \{\, \lambda \in \mathbb{R}^6 \;:\; p < 0, \quad q < 0, \quad r > 0; \quad a, b \in \mathbb{R}; \quad 0 < x < 1 \,\},$$
$$\mathcal{F}^+ := \{\, \lambda \in \mathbb{R}^6 \;:\; p > r, \quad q > r, \quad r > 0; \quad a, b \in \mathbb{R}; \quad 0 < x < 1 \,\},$$

where we have only to deal with $\mathcal{F}^-$, because Euler's transformation (16b) swaps $\mathcal{F}^-$ and $\mathcal{F}^+$. As is indicated in Figures 3 and 4, the reciprocity induces the following transpositions:

$$\mathcal{D}^- \leftrightarrow \mathcal{F}^-, \qquad \mathcal{E}^{*-} \leftrightarrow \mathcal{E}^{-*}.$$

A chief idea underlying duality and reciprocity is the concept of an Ebisu symmetry to be developed in §4, based on which these two symmetries will be constructed in §5 and §6.

3 Main Results

The main results of this article are established on the foundation of our previous work [8]. So this section begins with a review of our previous results, upon rearranged somewhat in a manner suitable for the purposes of the present as well as forthcoming papers.

3.1 Review of the Previous Results

As to Problem 1.1, Theorem 2.3 of [8] contains the following statement.

Theorem 3.1 *In region* $\mathcal{D} \cup \mathcal{I} \cup \mathcal{E}$ *any solution to Problem* $\mathbb{I}$ *leads back to a solution to Problem* I *and hence Problems* I *and* $\mathbb{I}$ *are equivalent.*

Thus in this region we can speak of a solution without specifying to which problem it is a solution. The next results are about solutions on the square domain $\mathcal{D}$, which are a collection of some parts of Theorems 2.1, 2.3, 2.4, 2.7 and 2.10, as well as of Remark 2.5 in [8].

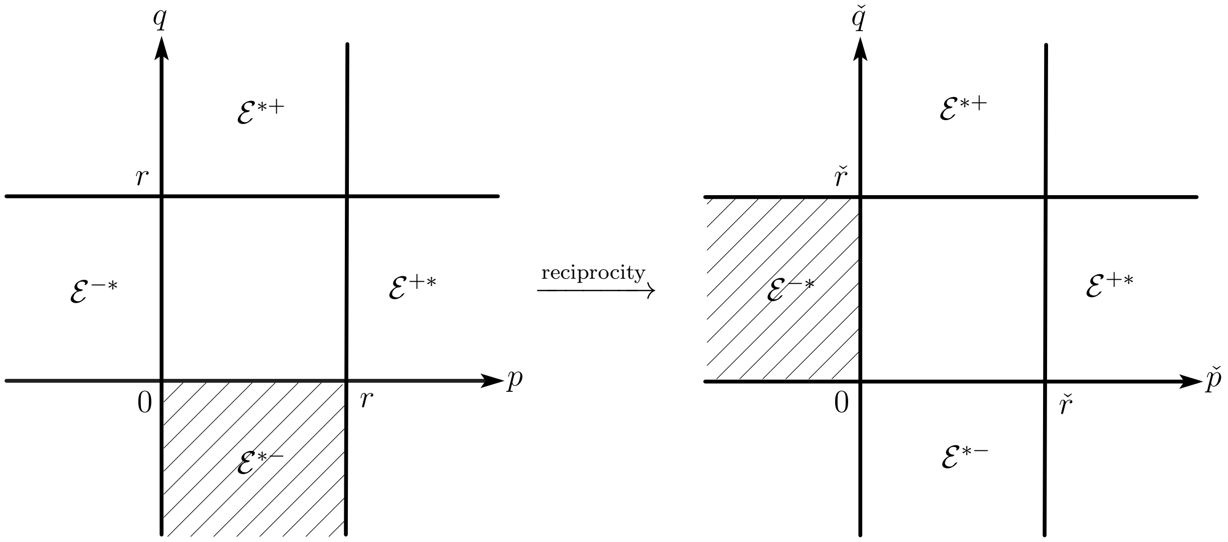

Figure 4: Reciprocity between $\mathcal{E}^{*-}$ and $\mathcal{E}^{-*}$.

Theorem 3.2 *The following hold in $\mathcal{D}$. Let $\lambda = (p, q, r; a, b; x) \in \mathcal{D}$ in what follows.*

(1) *Every solution in $\mathcal{D}^0$ is elementary, while every solution in $\mathcal{D}^{\pm}$ is non-elementary. All elementary solutions λ are at the center $\bullet$ of the square $\mathcal{D}$ in Figure* 1*, or more precisely,*

$$p = q = \frac{r}{2} > 0; \quad a = i, \quad b = j - \frac{1}{2}, \quad i, j \in \mathbb{Z}; \quad 0 < x < 1,$$

where a and b are exchangeable by symmetry while r and x are free. The corresponding $f(w; \lambda)$ is a degenerate hypergeometric functions with a dihedral monodromy group,

$$f(w) = S_{ij}(rw; x) \cdot \left(\tfrac{1+\sqrt{1-x}}{2}\right)^{1-rw},$$

studied by Vidunas [13, Theorem 3.1], where $S_{ij}(w; x)$ is a rational function of w explicitly representable in terms of terminating Appell's F_3 series (see Iwasaki [8, Theorem 2.1]).

(2) *Any solution $\lambda \in \mathcal{D}^{\pm}$ falls into one of the following two types:*

(A) *p, q, $r \in \mathbb{Z}$ and $0 \neq r - p - q \equiv 0 \mod 2$,*

(B) *p, $q \in \frac{1}{2} + \mathbb{Z}$ and $r \in \mathbb{Z}$.*

(3) *Any solution of type* (A) *in $\mathcal{D}^{\pm}$ comes from contiguous relations.*

(4) *If $\lambda \in \mathcal{D}^{\pm}$ is a solution of type* (B) *then its duplication $2\lambda := (2p, 2q, 2r; a, b; x) \in \mathcal{D}^{\pm}$ is an* (A)*-solution, so any* (B)*-solution in $\mathcal{D}^{\pm}$ essentially comes from contiguous relations.*

In [8, Theorem 2.4] we included certain additional conditions involving (a, b) to describe the dichotomy of types (A) and (B). Those conditions are omitted here because they will not be used in this article. In any case assertion (2) above implies that the (p, q, r)-component of any (A)-solution $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^-$ must belong to the integer domain

$$D_{\mathrm{A}}^- := \{\, \boldsymbol{p} = (p, q; r) \in \mathbb{Z}^3 \;:\; p > 0,\; q > 0,\; 0 < r - p - q \equiv 0 \mod 2 \,\}. \tag{22}$$

In view of assertion (4) above we may concentrate our attention on (A)-solutions. Then by assertion (3) the method of contiguous relations developed in [8, §11] produces the following results about the numbers x and the gamma product formulas (GPFs) for (A)-solutions.

Theorem 3.3 *Let $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^-$ be any* (A)*-solution with $\boldsymbol{p} = (p, q; r) \in D_{\mathrm{A}}^-$.*

(1) *x must be an algebraic number as a real root in $0 < z < 1$ of an algebraic equation*

$$Y(z) = Y(z; \boldsymbol{p}) = 0, \tag{23}$$

where $Y(z) \in \mathbb{Z}[w]$ depends only on $\boldsymbol{p}$ and is defined implicitly as follows: If we put

$$\varDelta(z) := (p-q)^2 z^2 - 2\{(p+q)r - 2pq\}z + r^2,$$
$$Z_{\pm}(z) := \left\{r + (p-q)z \pm \sqrt{\varDelta}\right\}^p \left\{r - (p-q)z \pm \sqrt{\varDelta}\right\}^q \left\{(2r-p-q)z - r \mp \sqrt{\varDelta}\right\}^{r-p-q},$$

then there exist unique polynomials $X(z)$, $Y(z) \in \mathbb{Z}[z]$ such that

$$Z_{\pm}(z) = X(z) \pm Y(z)\sqrt{\varDelta}.$$

(2) *There exists a positive constant $C > 0$ such that a gamma product formula*

$$f(w; \lambda) = C \cdot d^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(w + \frac{i}{r}\right)}{\prod_{i=1}^{r} \Gamma\left(w + v_i\right)} \tag{24}$$

holds true, where the number d is given by

$$d = \frac{r^r}{\sqrt{p^p q^q (r-p)^{r-p} (r-q)^{r-q} x^r (1-x)^{p+q-r}}}, \tag{25}$$

while $v_1, \dots, v_r$ are such numbers that sum up to

$$v_1 + \cdots + v_r = \frac{r-1}{2}, \tag{26}$$

and that admit a division relation in $\mathbb{C}[w]$,

$$\prod_{i=1}^{r}(w + v_i) \;\Big|\; \prod_{i=1}^{p-1}\left(w + \tfrac{i+a}{p}\right) \prod_{i=1}^{q-1}\left(w + \tfrac{i+b}{q}\right) \prod_{j=0}^{r-p-1}\left(w + \tfrac{j-a}{r-p}\right) \prod_{j=0}^{r-q-1}\left(w + \tfrac{j-b}{r-q}\right). \tag{27}$$

Here algebraic equation (23) stems from [8, assertion (1) of Theorem 2.10]. For any triple $\boldsymbol{p} = (p, q; r) \in D_{\mathrm{A}}^-$ the equation (23) has at least one root in the interval $0 < z < 1$ by [8, assertion (2) of Lemma 11.7]. Formula (25) comes from [8, formula (20) in Theorem 2.4]; condition (27) is from [8, assertion (3) of Theorem 2.10], whereas formula (24) follows from a combination of [8, assertion (3) of Theorem 2.10] and [8, Proposition 5.4] by taking $S(w) \equiv 1$, $m = r$ and $u_i = (i-1)/r$ ($i = 1, \dots, r$) in [8, formula (57)]. Condition (26) is derived by adapting the equality $v := v_1 + \cdots + v_r = u + s_0$ in [8, Proposition 5.4] to the current situation where $u := u_1 + \cdots + u_r = \sum_{i=1}^{r}(i-1)/r = (r-1)/2$ and $s_0 = 0$. In GPF (24) some but not all gamma factors in the numerator may coincide pairwise with (and hence can be canceled by) the same number of gamma factors in the denominator.

Article [8] mainly concerns the domain $\mathcal{D}$, but it also contains a few results on $\mathcal{I}$ and $\mathcal{E}$.

Theorem 3.4 *The following hold in $\mathcal{I}^{*-}$. Let $\lambda = (p, q = 0, r; a, b; x) \in \mathcal{I}^{*-}$ in what follows.*

(1) *λ is an elementary solution if and only if $b \in \mathbb{Z}_{\le 0}$, in which case the hypergeometric series that defines $f(w; \lambda)$ is terminating, so that $f(w; \lambda)$ itself is a rational function of w.*

(2) *For any non-elementary solution $\lambda \in \mathcal{I}^{*-}$ we have p, $r \in \mathbb{Z}$, $p \equiv r \mod 2$ and $b = 1/2$.*

(3) *Any non-elementary solution comes from contiguous relations.*

Here assertion (1) comes from [8, Theorem 2.1]; assertion (2) follows from a combination of [8, Theorem 2.3], [8, formula (80b) in Proposition 8.1] and [8, Lemma 10.1], while assertion (3) stems from [8, Proposition 11.2]. It is remarkable that one must have $b = 1/2$ for any non-elementary solution in $\mathcal{I}^{*-}$. Certainly all previously known examples enjoy this condition and this empirical fact has now been established logically and theoretically.

Proposition 3.5 *The following hold in $\mathcal{E}^{*-}$. Let $\lambda = (p, q, r; a, b; x) \in \mathcal{E}^{*-}$ in what follows.*

(1) *Every solution in $\mathcal{E}^{*-}$ is non-elementary.*

(2) *For any solution $\lambda \in \mathcal{E}^{*-}$ we must have p, $r \in \mathbb{Z}$ and $p \equiv r \mod 2$.*

(3) *Any solution $\lambda \in \mathcal{E}^{*-}$ with $q \in \mathbb{Z}$ comes from contiguous relations.*

Here assertion (1) comes from [8, Theorem 2.1]; assertion (2) follows from a combination of [8, Theorem 2.3], [8, formula (80c) in Proposition 8.1] and [8, Lemma 10.1], while assertion (3) stems from [8, Proposition 11.2]. It is not yet known whether $\mathcal{E}^{*-}$ contains a solution with non-integral or irrational $q \in \mathbb{R}$; for this reason Proposition 3.5 is not called Theorem 3.5.

3.2 Statement of the New Results

We proceed to stating the new results of this article. Most of them are results on domain $\mathcal{D}^-$ and are a substantial sharpening of the results in the previous article [8], becoming more arithmetic, while the remaining ones are on the new domain $\mathcal{F}^-$ which is focused on for the first time in this article. We also include a small but important new result on domain $\mathcal{E}^{*-}$. All these are largely indebted to the new ingredients of this article, that is, duality and reciprocity.

The first result is about the arithmetic properties of $(p, q; r)$ for (A)-solutions in $\mathcal{D}^-$.

Theorem 3.6 *For any* (A)*-solution $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^-$ the integer triple $\boldsymbol{p} = (p, q; r)$ must belong to D_{A}^- and moreover satisfy the division relations:*

$$(p|r \quad or \quad p|(r-p-q)) \quad and \quad (q|r \quad or \quad q|(r-p-q)). \tag{28}$$

This theorem does not refer to which option comes true in each "or" phrase of condition (28), while item (1) of Theorem 3.8 below has something to do with this question. Theorem 3.6 is presented at the beginning because it concerns the principal part $\boldsymbol{p}$ of the data λ, but its proof will be given rather late in §8 (Proposition 8.3), preceded by the proofs of Theorems 3.7, 3.8, 3.10 and 3.11 below. The second result is regarding the duality for (A)-solutions in $\mathcal{D}^-$.

Theorem 3.7 *The duality $\lambda \mapsto \lambda'$ defined by* (18) *yields an involution on the set of all* (A)*-solutions in $\mathcal{D}^-$. For those solutions it induces a transformation of gamma product formulas:*

$$f(w;\lambda) = C \cdot d^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(w+\frac{i}{r}\right)}{\prod_{i=1}^{r} \Gamma\left(w+v_i\right)} \quad \mapsto \quad f(w;\lambda') = C' \cdot d^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(w+\frac{i}{r}\right)}{\prod_{i=1}^{r} \Gamma\left(w+v_i'\right)}, \tag{29}$$

where C' is a positive constant, d is the number given by formula (25)*, while*

$$v_i' := 1 - \frac{2}{r} - v_i^* \qquad (i = 1, \dots, r), \tag{30}$$

with $v_1^*, \dots, v_r^*$ being the numbers complementary to $v_1, \dots, v_r$, to the effect that

$$\prod_{i=1}^{r}(w+v_i)\prod_{i=1}^{r}(w+v_i^*) = \prod_{i=0}^{p-1}\left(w+\tfrac{i+a}{p}\right)\prod_{i=0}^{q-1}\left(w+\tfrac{i+b}{q}\right)\prod_{j=0}^{r-p-1}\left(w+\tfrac{j-a}{r-p}\right)\prod_{j=0}^{r-q-1}\left(w+\tfrac{j-b}{r-q}\right). \tag{31}$$

Notice that condition (31) is well defined, since we have division relation (27). The proof of Theorem 3.7 will be given in §5 (see Proposition 5.4). Notice also that the first and second products on the right-hand side of division relation (27) are taken from $i = 1$, whereas those of equation (31) are from $i = 0$. This subtle difference eventually grows into the following significant result about the arithmetic nature of (a, b) as well as its dual (a', b').

Theorem 3.8 *For any* (A)*-solution $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^-$ and its dual* (A)*-solution $\lambda' = (p, q, r; a', b'; x) \in \mathcal{D}^-$, the pair (λ, λ') must be subject to the conditions listed in Table* 1*, which consist of the following four items:*

(1) *two division relations for the integer triple $\boldsymbol{p} = (p, q; r)$,*

(2) *a formula for (a, b) in terms of $\boldsymbol{p}$ and a pair of nonnegative integers (i, j),*

(3) *a formula for (a', b') in terms of $\boldsymbol{p}$ and a pair of nonnegative integers (i', j'),*

(4) *two $\mathbb{Z}$-linear equations for quadruple $(i, j; i', j') \in \mathbb{Z}_{\geq 0}^4$,*

where for each $\nu \in \{1, 2, 3, 4\}$ item (ν) is exhibited in column (ν) of Table 1*. There are a total of six cases or patterns, up to equivalence, according to the type of the dual pair (λ, λ') shown in column* (5)*, the notion of which will be defined in* §7 *and may be skipped until there. Moreover,*

$$a, b, a', b', v_1, \dots, v_r, v_1', \dots, v_r' \in \mathbb{Q} \cap [0, 1), \tag{32}$$

where $v_1, \dots, v_r$ and $v_1', \dots, v_r'$ are the numbers appearing in formula (29)*.*

Theorem 3.8 will be proved in §7 (see Proposition 7.5) after a preliminary discussion in §6.

Remark 3.9 A few remarks about Table 1 should be in order at this stage.

(1) In Table 1 (and also in §7–§8), for positive integers s and t we write $s_t := s/t$ when and only when $t|s$, i.e., t divides s. It is a convenient notation which indicates at once that s_t stands for a positive integer; it also saves space when s is a large expression, but for example $(r - p - q)_p$ should not be confused with a rising factorial number.

	(1)	(2)	(3)	(4)	(5)
case	division relations	formula for (a,b)	formula for (a',b')	$\mathbb{Z}$-linear equations for $(i,j;i',j')$	type
1	$p\|r$ $q\|r$	$a=\frac{i}{r_p}$ $b=\frac{j}{r_q}$	$a'=\frac{i'}{r_p}$ $b'=\frac{j'}{r_q}$	$i+i'=r_p-2$ $j+j'=r_q-2$	I I I I
2	$p\|(r-p-q)$ $q\|(r-p-q)$	$a=\frac{(r-p)i-qj}{r(r-p-q)_p}$ $b=\frac{(r-q)j-pi}{r(r-p-q)_q}$	$a'=\frac{(r-p)i'-qj'}{r(r-p-q)_p}$ $b'=\frac{(r-q)j'-pi'}{r(r-p-q)_q}$	$i+i'=(r-p-q)_p$ $j+j'=(r-p-q)_q$	II II II II
3	$p\|r$ $q\|(r-p-q)$	$a=\frac{i}{r_p}$ $b=\frac{r_pj-i}{r_p(r-p)_q}$	$a'=\frac{i'}{r_p}$ $b'=\frac{r_pj'-i'}{r_p(r-p)_q}$	$i+i'=r_p-2$ $j+j'=(r-p-q)_q$	I II I II
4	$p\|r$ $q\|r_p(r-p-q)$	$a=\frac{i}{r_p}$ $b=\frac{qj}{r}$	$a'=\frac{i'}{r_p}$ $b'=\frac{r_pj'-i'}{(r_p(r-p))_q}$	$i+i'=r_p-2$ $i+(r_p-1)j+r_pj'=(r_p(r-p-q))_q$	I I I II
5	$p\|(r-p-q)$ $q\|r(r-p-q)_p$	$a=\frac{(r-p)i-qj}{r(r-p-q)_p}$ $b=\frac{(r-q)_pj-i}{(r(r-p-q)_p)_q}$	$a'=\frac{ri'-qj'}{r(r-q)_p}$ $b'=\frac{qj'}{r}$	$i+i'=(r-p-q)_p$ $i'+(r-q)_pj+(r-p-q)_pj'=((r-q)(r-p-q)_p)_q$	II II II I
6	$p\|r(r-p-q)$ $q\|r(r-p-q)$	$a=\frac{pi}{r}$ $b=\frac{rj-pi}{(r(r-p))_q}$	$a'=\frac{ri'-qj'}{(r(r-q))_p}$ $b'=\frac{qj'}{r}$	$(r-p-q)i+(r-p)i'+qj=((r-p)(r-p-q))_p$ $(r-p-q)j'+pi'+(r-q)j=((r-q)(r-p-q))_q$	I II II I

Table 1: Candidates for dual pairs of (A)-solutions in $\mathcal{D}^-$.

(2) The division relations in column (1) have something to do with those in condition (28). Indeed, in case 1 the divisions $p|r$ and $q|r$ must occur in the two "or" phrases in (28). Cases 2 and 3 are similar to case 1, that is, there are unique choices in the two "or"s. In case 4 the division $p|r$ must happen in the former "or", while either option in the latter "or" guarantees the condition $q|r_p(r-p-q)$. Case 5 is similar to case 4. In case 6 any combination in the two "or"s implies $p|r(r-p-q)$ and $q|r(r-p-q)$.

(3) In each case of Table 1 the $\mathbb{Z}$-linear equations for $(i,j;i',j') \in \mathbb{Z}^4_{\geq 0}$ are of the form

$$\mu_1 i + \mu_2 j + \mu_3 i' + \mu_4 j' = \mu_5, \qquad \nu_1 i + \nu_2 j + \nu_3 i' + \nu_4 j' = \nu_5, \tag{33}$$

where for each $k \in \{1,2,3,4\}$ the coefficients μ_k and ν_k are nonnegative integers with a positive sum $\mu_k + \nu_k \geq 1$ so that system (33) cannot have infinitely many solutions.

A data $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ with $\boldsymbol{p} = (p,q;r) \in D^-_{\mathrm{A}}$ is said to be a *candidate* for an (A)-solution, if $\boldsymbol{p}$ satisfies condition (28) and λ is subject to one of the six patterns in Table 1. By Theorems 3.6 and 3.8, a data cannot be an (A)-solution unless it is a candidate, but it may (perhaps quite frequently) happen that a candidate is not an actual (A)-solution.

Theorem 3.8 together with assertion (1) of Theorem 3.3 gives the following.

Theorem 3.10 *For any* (A)*-solution* $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ *the numbers* a *and* b *must be rational while* x *must be algebraic. Given any integer triple* $\boldsymbol{p} = (p,q;r) \in D^-_{\mathrm{A}}$*, there are no or only a finite number of* (A)*-solutions* $\lambda \in \mathcal{D}^-$ *with the prescribed principal data* $\boldsymbol{p}$*.*

We turn our attention to the reciprocity $\lambda \mapsto \check{\lambda}$ defined by formula (20). Considering the reciprocals of (A)-solutions on $\mathcal{D}^-$ not only produces new solutions on the target domain $\mathcal{F}^-$, but also brings us a deeper understanding of the original solutions on the source domain $\mathcal{D}^-$.

Theorem 3.11 *For any* (A)*-solution* $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ *there exists a division relation*

$$\prod_{i=0}^{p-1} \left(w + \tfrac{i+a}{p}\right) \prod_{i=0}^{q-1} \left(w + \tfrac{i+b}{q}\right) \Bigm| \prod_{i=1}^{r} (w + v_i) \qquad \text{in} \quad \mathbb{Q}[w], \tag{34}$$

which allows us to rearrange the numbers $v_1, \dots, v_r$ *in formula* (24) *so that*

$$\prod_{i=0}^{p-1} \left(w + \tfrac{i+a}{p}\right) \prod_{i=0}^{q-1} \left(w + \tfrac{i+b}{q}\right) = \prod_{i=r-p-q+1}^{r} (w + v_i)\,. \tag{35}$$

With this convention the reciprocity $\lambda \mapsto \check{\lambda}$ *sends any* (A)*-solution* $\lambda \in \mathcal{D}^-$ *to a new solution* $\check{\lambda}$ *to Problem* I *in* $\mathcal{F}^-$*, inducing a transformation of gamma product formulas:*

$$f(w;\lambda) = C \cdot d^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(w + \frac{i}{r}\right)}{\prod_{i=1}^{r} \Gamma(w + v_i)} \;\mapsto\; f(w;\check{\lambda}) = \check{C} \cdot \check{d}^w \cdot \frac{\prod_{i=0}^{r-p-q-1} \Gamma\left(w + \frac{i}{r-p-q}\right)}{\prod_{i=1}^{r-p-q} \Gamma(w + \check{v}_i)}, \tag{36}$$

where $\check{C}$ *is a positive constant and*

$$\check{d} = (r-p-q)^{r-p-q} \sqrt{\frac{p^p\, q^q\, x^r}{(r-p)^{r-p}\,(r-q)^{r-q}\,(1-x)^{r-p-q}}}, \tag{37}$$

$$\check{v}_i = v_i - \frac{1-a-b}{r-p-q} \qquad (i = 1, \dots, r-p-q). \tag{38}$$

Theorem 3.11 will be proved in §7 as a main part of Proposition 7.6. This theorem concerns the reciprocity in the direction $\mathcal{D}^- \to \mathcal{F}^-$. Starting from $\mathcal{F}^-$ take the reciprocity in the other way round $\mathcal{F}^- \to \mathcal{D}^-$ and then use Theorem 3.11 to return home in $\mathcal{F}^-$. This idea leads to the following solutions to Problems 1.1–1.5 on the domain $\mathcal{F}^-$.

Theorem 3.12 *The following hold on $\mathcal{F}^-$. Let $\lambda = (p, q, r; a, b; x) \in \mathcal{F}^-$ in what follows.*

(1) *Any solution $\lambda \in \mathcal{F}^-$ to Problem* I *or* II *is non-elementary.*

(2) *If $\lambda \in \mathcal{F}^-$ is an integral solution to Problem* II*, then λ comes from contiguous relations with its reciprocal $\check{\lambda}$ being an* (A)*-solution in $\mathcal{D}^-$, in other words, λ is the reciprocal of an* (A)*-solution $\check{\lambda} \in \mathcal{D}^-$ so that r must be an even positive integer and λ becomes a solution to Problem* I *with $f(w; \lambda)$ admitting a GPF as stated in assertion* (3) *below.*

(3) *If $\lambda \in \mathcal{F}^-$ is a rational solution to Problem* II*, then λ essentially comes from contiguous relations, r must be a positive integer but not necessarily even; a, $b \in \mathbb{Q}$; x algebraic, and λ becomes a solution to Problem* I *with $f(w; \lambda)$ having a GPF of the form*

$$f(w; \lambda) = C \cdot d^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(w + \frac{i}{r}\right)}{\prod_{i=1}^{r} \Gamma\left(w + v_i\right)}, \tag{39}$$

where C is a positive constant, d is a positive algebraic number defined by

$$d = r^r \sqrt{\frac{|p|^{|p|}\, |q|^{|q|}\, (1-x)^{r-p-q}}{(r-p)^{r-p}\, (r-q)^{r-q}\, x^r}}, \tag{40}$$

and $v_1, \dots, v_r$ are such numbers that satisfy the following conditions:

$$v_1 + \cdots + v_r = \frac{r-1}{2}, \qquad v_1, \dots, v_r \in \left(\mathbb{Q} \setminus \tfrac{1}{r}\mathbb{Z}\right) \cap [c,\, c+1), \qquad c := \frac{1-a-b}{r-p-q}. \tag{41}$$

In particular, Problems I *and* II *are equivalent for rational data in domain $\mathcal{F}^-$.*

Remark 3.13 We make three comments about Theorem 3.12.

(1) When λ is integral, formula (39) is computable from the GPF for the (A)-solution $\check{\lambda} \in \mathcal{D}^-$ through transformation (36), where the roles of λ and $\check{\lambda}$ are exchanged. Regarding duality it is possible to formulate a result similar to Theorem 3.7 for integral solutions in $\mathcal{F}^-$.

(2) We mention how to calculate formula (39) when λ is rational but not integral. If k is the least common denominator of p, $q \in \mathbb{Q}$, then $k\lambda \in \mathcal{F}^-$ is an integral solution to Problem II, so that as in item (1) the GPF for $k\lambda$ is computable from the GPF for the (A)-solution $(k\lambda)^\vee \in \mathcal{D}^-$ via transformation (36). It turns out that the result is

$$f(w; k\lambda) = C_k \cdot d^{kw} \cdot \frac{\prod_{i=0}^{kr-1} \Gamma\left(w + \frac{i}{kr}\right)}{\prod_{i=1}^{r} \prod_{j=0}^{k-1} \Gamma\left(w + \frac{v_i + j}{k}\right)}, \tag{42}$$

for some $C_k > 0$ and $v_1, \dots, v_r$ satisfying condition (41). Replacing w with w/k in formula (42) and using the multiplication formula (12) for the gamma function we get the desired formula (39). For details we refer to the proofs of Lemma 9.5 and Proposition 9.6.

(3) Theorem 3.12 concerns only rational solutions in $\mathcal{F}^-$. It is an interesting open problem to know whether $\mathcal{F}^-$ contains any irrational solution (see Problem 11.4).

Assertions (1) and (2) of Theorem 3.12 will be proved in §9 as Propositions 9.4, while assertion (3) will be established as Proposition 9.6 by developing the idea in item (2) of Remark 3.13. This idea for rational solutions in $\mathcal{F}^-$ can also be used to get information about the GPFs for (B)-solutions in $\mathcal{D}^-$ because the latter situation is similar to the former or even simpler in the sense that only duplication $k = 2$ is involved for the latter class of solutions.

Theorem 3.14 *If $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^-$ is a* (B)*-solution then a and b are rational numbers with $0 \le a,\ b < 1$, x is an algebraic number, and λ admits a GPF of the form* (24) *where d is given by formula* (25) *and $v_1, \dots, v_r$ are rational numbers such that $0 \le v_1, \dots, v_r < 1$.*

Only a hint for the proof of this theorem will be given in §9 after the proof of Proposition 9.6. In §10 we shall put all the results on $\mathcal{D}^- \cup \mathcal{F}^-$ into context from an algorithmic point of view and discuss how to find all solutions λ with a given bound for r in a finite number of steps.

Finally we present a small but important result on the domain $\mathcal{E}^{*-} \cup \mathcal{E}^{-*}$, where Problems I and II are equivalent by Theorem 3.1, so we can speak of a solution without specifying to which problem it is a solution. By assertion (2) of Proposition 3.5, for any solution $\lambda = (p, q, r; a, b; x) \in \mathcal{E}^{*-}$ we have $p,\ r \in \mathbb{Z}$ so that λ is integral if and only if $q \in \mathbb{Z}$. Similarly, for any solution $\lambda = (p, q, r; a, b; x) \in \mathcal{E}^{-*}$ we have $q,\ r \in \mathbb{Z}$, so that λ is integral if and only if $p \in \mathbb{Z}$. It is not known whether $q \in \mathbb{Z}$ resp. $p \in \mathbb{Z}$ for every solution $\lambda \in \mathcal{E}^{*-}$ resp. $\lambda \in \mathcal{E}^{-*}$.

Theorem 3.15 *Duality $\lambda \mapsto \lambda'$ in* (18) *induces a self-bijection on the set of all integral solutions in $\mathcal{E}^{*-}$. Similarly, reciprocity $\lambda \mapsto \check{\lambda}$ in* (20) *induces a bijection between the set of all integral solutions in $\mathcal{E}^{*-}$ and the set of all integral solutions in $\mathcal{E}^{-*}$.*

The former and latter assertions of this theorem will be proved in §5 and §6 as Corollaries 5.2 and 6.2 respectively. Applications of the theorem will be discussed elsewhere.

4 Kummer's 24 Solutions and Ebisu Symmetries

Kummer [10] constructed twenty-four solutions (or more precisely power series representations of solutions) to the Gauss hypergeometric equation (1). They are known as *Kummer's twenty-four solutions*, among which the hypergeometric series ${}_2F_1(\boldsymbol{a}; z)$ is the most representative member. A complete list of them can be found in Erdélyi [6, Chap. II, §2.9, formulas (1)–(24)]. Ebisu [5, Lemma 3.2.2] showed that each of Kummer's solutions, say ${}_2K_1(\boldsymbol{a}; z)$, admits a three-term relation of the following form: for every integer vector $\boldsymbol{p} = (p, q; r) \in \mathbb{Z}^3$,

$$
{}_2K_1(\boldsymbol{a} + \boldsymbol{p}; z) = \psi(\boldsymbol{a}; \boldsymbol{p})\, r(\boldsymbol{a}; z)\, {}_2K_1(\boldsymbol{a}; z) + \phi(\boldsymbol{a}; \boldsymbol{p})\, q(\boldsymbol{a}; z)\, {}_2K_1(\boldsymbol{a} + \mathbf{1}; z), \tag{43}
$$

where $q(\boldsymbol{a}; z)$ and $r(\boldsymbol{a}; z)$ are the same functions as those in the original three-term relation (8) for ${}_2F_1(\boldsymbol{a}; z)$, whereas $\phi(\boldsymbol{a}; \boldsymbol{p})$ and $\psi(\boldsymbol{a}; \boldsymbol{p})$ are nontrivial rational functions of $\boldsymbol{a}$ depending uniquely on ${}_2K_1(\boldsymbol{a}; z)$ and $\boldsymbol{p}$, explicit formulas for which can be found in [5].

Problem II makes sense not only for ${}_2F_1(\boldsymbol{a}; z)$ but also for any other member ${}_2K_1(\boldsymbol{a}; z)$ of Kummer's solutions. Given an integral data $\lambda = (p, q, r; a, b; x)$, if we put $k(w; \lambda) :=$

${}_2K_1(\boldsymbol{\alpha}(w);x)$, $\tilde{k}(w;\lambda) := {}_2K_1(\boldsymbol{\alpha}(w)+\mathbf{1};x)$, $\Phi(w;\lambda) := \phi(\boldsymbol{\alpha}(w);x)$ and $\Psi(w;\lambda) := \psi(\boldsymbol{\alpha}(w);x)$ with $\boldsymbol{\alpha}(w) := (pw+a,\, qw+b;\, rw)$, then the three-term relation (43) leads to

$$k(w+1;\lambda) = \Psi(w;\lambda)R(w;\lambda)\cdot k(w;\lambda) + \Phi(w;\lambda)Q(w;\lambda)\cdot \tilde{k}(w;\lambda), \tag{44}$$

just as the relation (8) leads to (9). Notice that $\Phi(w;\lambda)$ and $\Psi(w;\lambda)$ are nontrivial rational functions of w. This observation gives the following important lemma.

Lemma 4.1 *Let ${}_2K_1(\boldsymbol{a};z)$ be any member of Kummer's twenty-four solutions. An integral data λ is a solution to Problem II for the function ${}_2F_1(\boldsymbol{a};z)$ that comes from contiguous relations, if and only if the same is true for the function ${}_2K_1(\boldsymbol{a};z)$. If this is the case, then*

$$\frac{k(w+1;\lambda)}{k(w;\lambda)} = \Psi(w;\lambda)\cdot R(w;\lambda) \in \mathbb{C}(w), \tag{45}$$

which is corresponding to condition (6) *for the original function $f(w;\lambda)$.*

Proof. Recall that an integral data λ is a solution to Problem II for ${}_2F_1(\boldsymbol{a};z)$ that comes from contiguous relations if and only if condition (10) is satisfied. In view of formula (44) the corresponding condition for ${}_2K_1(\boldsymbol{a};z)$ is $\Phi(w;\lambda)Q(w;\lambda)=0$ in $\mathbb{C}(w)$. But this is just equivalent to condition (10), because $\Phi(w;\lambda)$ is nontrivial. Now formula (44) leads to condition (45). $\square$

Remark 4.2 Any member ${}_2K_1(\boldsymbol{a};z)$ of Kummer's 24 solutions can be written ${}_2K_1(\boldsymbol{a};z) =$ (an elementary factor) $\times\, {}_2F_1(\tilde{\boldsymbol{a}};\tilde{z})$ in terms of the original hypergeometric function ${}_2F_1(\boldsymbol{a};z)$ and a certain transformation of variables $(\boldsymbol{a};z)\mapsto(\tilde{\boldsymbol{a}};\tilde{z})$. So Lemma 4.1 suggests that each ${}_2K_1(\boldsymbol{a};z)$ brings a symmetry to Problem II for the original function ${}_2F_1(\boldsymbol{a};z)$. It may be referred to as an *Ebisu symmetry* because it originates from Ebisu's observation (43). The existence of Ebisu symmetries is an advantage of dealing with Problem II, whereas such a helpful structure cannot be expected for Problem I, although we must keep it in mind that *Ebisu symmetries make sense only on those solutions which come from contiguous relations*.

In this article we exhibit two special choices of Kummer's solutions other than the original one ${}_2F_1(\boldsymbol{a};z)$. The resulting Ebisu symmetries will be the main players in this article, that is, duality and reciprocity. Here one choice of ${}_2K_1(\boldsymbol{a};z)$ is to take

$${}_2G_1(\boldsymbol{a};z) := z^{1-\gamma}{}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z), \tag{46}$$

which is the solution of local exponent $1-\gamma$ at $z=0$ in Riemann scheme (2). Note that solutions ${}_2F_1(\boldsymbol{a};z)$ and ${}_2G_1(\boldsymbol{a};z)$ form a linear basis of local solutions to the hypergeometric equation (1) at $z=0$, unless γ is an integer. The other choice is

$${}_2H_1(\boldsymbol{a};z) := z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}{}_2F_1(1-\alpha,1-\beta;\gamma-\alpha-\beta+1;1-z), \tag{47}$$

which is an expression for the local solution of exponent $\gamma-\alpha-\beta$ at $z=1$ in the scheme (2).

When ${}_2K_1(\boldsymbol{a};z)$ is ${}_2G_1(\boldsymbol{a};z)$ or ${}_2H_1(\boldsymbol{a};z)$, we use Lemma 4.1 to construct duality or reciprocity, where we employ the following notation. For ${}_2K_1(\boldsymbol{a};z) = {}_2G_1(\boldsymbol{a};z)$ the functions $k(w;\lambda)$, $\psi(\boldsymbol{a};\boldsymbol{p})$ and $\Psi(w;\lambda)$ are denoted by $g(w;\lambda)$, $\psi_g(\boldsymbol{a};\boldsymbol{p})$ and $\Psi_g(w;\lambda)$, while for ${}_2K_1(\boldsymbol{a};z) = {}_2H_1(\boldsymbol{a};z)$ they are denoted by $h(w;\lambda)$, $\psi_h(\boldsymbol{a};\boldsymbol{p})$ and $\Psi_h(w;\lambda)$, respectively. Note that

$$g(w;\lambda) = x^{1-rw}{}_2F_1((p-r)w+a+1,(q-r)w+b+1;2-rw;x), \tag{48}$$

$$\begin{aligned} h(w;\lambda) = {} & x^{1-rw}(1-x)^{(r-p-q)w-a-b} \\ & \times {}_2F_1(1-a-pw,\,1-b-qw;\,(r-p-q)w+1-a-b;\,1-x). \end{aligned} \tag{49}$$

From a result of Ebisu [5, Lemma 3.2.2] we have

$$\psi_g(\boldsymbol{a};\boldsymbol{p}) = (-1)^{r-p-q}\frac{(\alpha)_p(\beta)_q(\gamma-\alpha)_{r-p}(\gamma-\beta)_{r-q}}{(\gamma-1)_r(\gamma)_r},$$

$$\psi_h(\boldsymbol{a};\boldsymbol{p}) = (-1)^{r-p-q}\frac{(\alpha)_p(\beta)_q(\gamma-\alpha-\beta+1)_{r-p-q}}{(\gamma)_r},$$

in formula (43), from which we find

$$\Psi_g(w;\lambda) = (-1)^{r-p-q}\frac{(pw+a)_p(qw+b)_q((r-p)w-a)_{r-p}((r-q)w-b)_{r-q}}{(rw-1)_r(rw)_r}, \tag{50}$$

$$\Psi_h(w;\lambda) = (-1)^{r-p-q}\frac{(pw+a)_p(qw+b)_q((r-p-q)w-a-b+1)_{r-p-q}}{(rw)_r}, \tag{51}$$

in formula (45), where $(s)_n := \Gamma(s+n)/\Gamma(s)$ is Pochhammer's symbol or the rising factorial. Solution (46) and formula (50) will be used to construct duality in §5, while solution (47) and formula (51) will be employed to construct reciprocity in §6, respectively.

5 Duality

Applying Lemma 4.1 to ${}_2K_1(\boldsymbol{a};z) = {}_2G_1(\boldsymbol{a};z)$ leads to the duality (18) in Definition 2.1.

Lemma 5.1 *Let $\lambda = (p,q,r;a,b;x)$ be an integral data in domain (17). If λ is a solution to Problem II that comes from contiguous relations, with rational function $R(w;\lambda)$ in condition (6), then its dual $\lambda' = (p',q',r';a',b';x')$ is also a solution to Problem II that comes from contiguous relations, with the corresponding rational function*

$$R(w;\lambda') = \frac{x^{-r}(1-x)^{r-p-q}}{\Psi_g(w';\lambda)\,R(w';\lambda)}, \tag{52}$$

where the function $\Psi_g(w;\lambda)$ is given by formula (50) and $w \mapsto w'$ is the reflection

$$w' := \frac{2}{r} - 1 - w. \tag{53}$$

Proof. Replacing w by $w+1 = \frac{2}{r} - w'$ in formula (48), we observe that

$$\begin{aligned} g(w+1;\lambda) &= x^{1-r\left(\frac{2}{r}-w'\right)}{}_2F_1\left((p-r)\left(\tfrac{2}{r}-w'\right)+a+1,\, (q-r)\left(\tfrac{2}{r}-w'\right)+b+1; 2-r\left(\tfrac{2}{r}-w'\right); x\right) \\ &= x^{rw'-1}{}_2F_1\left((r-p)w'-a',\, (r-q)w'-b'; rw'; x\right) \\ &= x^{rw'-1}(1-x)^{(p+q-r)w'+a'+b'} f(w';\lambda'), \end{aligned}$$

where definition (18) and Euler's transformation (15b) are used in the second and third equalities respectively. Since the shift $w \mapsto w-1$ is equivalent to $w' \mapsto w'+1$, we have

$$\begin{aligned} f(w';\lambda') &= x^{1-rw'}(1-x)^{(r-p-q)w'-a'-b'} g(w+1;\lambda), \\ f(w'+1;\lambda') &= x^{1-r(w'+1)}(1-x)^{(r-p-q)(w'+1)-a'-b'} g(w;\lambda), \end{aligned} \tag{54}$$

which together with formula (45) in Lemma 4.1 yields

$$R(w';\lambda') := \frac{f(w'+1;\lambda')}{f(w';\lambda')} = x^{-r}(1-x)^{r-p-q}\cdot\frac{g(w;\lambda)}{g(w+1;\lambda)} = \frac{x^{-r}(1-x)^{r-p-q}}{\Psi_g(w;\lambda)\,R(w;\lambda)}.$$

Replacing w by w' in the above and noting $w''=w$, we obtain formula (52). □

Using Lemma 5.1 in domain $\mathcal{E}^{*-}$ leads to an immediate consequence.

Corollary 5.2 *Duality $\lambda\mapsto\lambda'$ in* (18) *induces a self-bijection on the set of all solutions $\lambda=(p,q,r;a,b;x)\in\mathcal{E}^{*-}$ with $q\in\mathbb{Z}$.*

Proof. In view of definition (18) the duality $\lambda\mapsto\lambda'$ is a bijection $\mathcal{E}^{*-}\to\mathcal{E}^{*-}$ in the data level. By assertion (3) of Proposition 3.5 and Lemma 5.1, it induces a bijection in the solution level among all solutions $\lambda\in\mathcal{E}^{*-}$ to Problem II (and to Problem I by Theorem 3.1) with $q\in\mathbb{Z}$. □

The same results holds true for (A)-solutions in $\mathcal{D}^-$ since the duality is also a bijection $\mathcal{D}^-\to\mathcal{D}^-$ in the data level, but in fact we are able to obtain more detailed results on $\mathcal{D}^-$.

Lemma 5.3 *If $\lambda=(p,q,r;a,b;x)\in\mathcal{D}^-$ is an* (A)*-solution to Problem II, then its dual $\lambda'=(p,q,r;a',b';x)\in\mathcal{D}^-$ is also an* (A)*-solution to the same problem with*

$$R(w;\lambda') = d\cdot\frac{\prod_{i=0}^{r-1}\left(w+\frac{i}{r}\right)}{\prod_{i=1}^{r}(w+v_i')}, \tag{55}$$

where d and $v_1',\dots,v_r'$ are defined by formulas (25) *and* (30) *respectively.*

Proof. By assertion (2) of Theorem 3.3 we have the gamma product formula (24), so that the rational function $R(w;\lambda)$ in formula (7) is given by

$$R(w;\lambda) = d\cdot\frac{\prod_{i=0}^{r-1}\left(w+\frac{i}{r}\right)}{\prod_{i=1}^{r}(w+v_i)}, \tag{56}$$

where d is defined in formula (25). On the other hand, formula (50) can be rewritten

$$\Psi_g(w;\lambda) = \frac{p^p q^q (r-p)^{r-p}(r-q)^{r-q}}{r^{2r}} \times\frac{\prod_{i=0}^{p-1}\left(w+\frac{i+a}{p}\right)\prod_{i=0}^{q-1}\left(w+\frac{i+b}{q}\right)\prod_{i=0}^{r-p-1}\left(w+\frac{i-a}{r-p}\right)\prod_{i=0}^{r-q-1}\left(w+\frac{i-b}{r-q}\right)}{\prod_{i=0}^{r-1}\left(w+\frac{i-1}{r}\right)\prod_{i=0}^{r-1}\left(w+\frac{i}{r}\right)},$$

where $(-1)^{r-p-q}=1$ is taken into account, which follows from assertion (2) of Theorem 3.2. Thus taking the product of equations (50) and (56) and using definition (31), we have

$$\frac{1}{\Psi_g(w;\lambda)\,R(w;\lambda)} = d\cdot x^r\cdot(1-x)^{p+q-r}\cdot\frac{\prod_{i=0}^{r-1}\left(w+\frac{i-1}{r}\right)}{\prod_{i=1}^{r}(w+v_i^*)}.$$

Replacing w by w' in the above, where w' is defined by (53), and using $\prod_{i=0}^{r-1}\left(w'+\frac{i-1}{r}\right)=(-1)^r\prod_{i=0}^{r-1}\left(w+\frac{i}{r}\right)$ and $\prod_{i=0}^{r-1}(w'+v_i^*)=(-1)^r\prod_{i=0}^{r-1}(w+v_i')$, we have

$$\frac{1}{\Psi_g(w';\lambda)\,R(w';\lambda)} = d\cdot x^r\cdot(1-x)^{p+q-r}\cdot\frac{\prod_{i=0}^{r-1}\left(w+\frac{i}{r}\right)}{\prod_{i=1}^{r}(w+v_i')}.$$

Substituting this into formula (52) leads to the desired formula (55). □

Proposition 5.4 *If $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^-$ is an* (A)*-solution to Problem* I *with gamma product formula* (24)*, then there exists a positive constant $C' > 0$ such that*

$$f(w; \lambda') = C' \cdot d^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(w + \frac{i}{r}\right)}{\prod_{i=1}^{r} \Gamma\left(w + v_i'\right)}, \tag{57}$$

$$g(w; \lambda) = D \cdot \delta^w \cdot \frac{\prod_{i=1}^{r} \sin\pi\left(w + v_i^*\right)}{\prod_{i=0}^{r-1} \sin\pi\left(w + \frac{i-1}{r}\right)} \cdot \frac{\prod_{i=1}^{r} \Gamma\left(w + v_i^*\right)}{\prod_{i=0}^{r-1} \Gamma\left(w + \frac{i-1}{r}\right)}, \tag{58}$$

where d, v_i' and v_i^* are given by formulas (25)*,* (30) *and* (31)*, whereas D and δ are defined by*

$$D := \frac{x \cdot d^{2/r} \cdot C'}{(1-x)^{a+b}}, \qquad \delta := \frac{(1-x)^{r-p-q}}{d \cdot x^r}. \tag{59}$$

Proof. By Lemma 5.3, λ' is a solution to Problem II in $\mathcal{D}^-$ with rational function $R(w; \lambda')$ in formula (55). Theorem 3.1 then implies that it leads back to a solution to Problem I, which is exactly the one in formula (57). Next we have

$$\begin{aligned}
g(w; \lambda) &= x^{r(w'+1)-1}(1-x)^{a'+b'-(r-p-q)(w'+1)}\, f(w'+1; \lambda') \\
&= \frac{x}{(1-x)^{a+b}} \cdot \left\{\frac{(1-x)^{r-p-q}}{x^r}\right\}^w \cdot f\left(\tfrac{2}{r} - w; \lambda'\right) \\
&= \frac{x}{(1-x)^{a+b}} \cdot \left\{\frac{(1-x)^{r-p-q}}{x^r}\right\}^w \cdot C' \cdot d^{\frac{2}{r}-w} \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(\frac{2}{r} - w + \frac{i}{r}\right)}{\prod_{i=1}^{r} \Gamma\left(\frac{2}{r} - w + v_i'\right)} \\
&= \frac{C' \cdot x \cdot d^{\frac{2}{r}}}{(1-x)^{a+b}} \cdot \left\{\frac{(1-x)^{r-p-q}}{d \cdot x^r}\right\}^w \frac{\prod_{i=0}^{r-1} \Gamma\left(1 - (w + \frac{(r-1-i)-1}{r})\right)}{\prod_{i=1}^{r} \Gamma\left(1 - (w + v_i^*)\right)} \\
&= D \cdot \delta^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(1 - (w + \frac{i-1}{r})\right)}{\prod_{i=1}^{r} \Gamma\left(1 - (w + v_i^*)\right)}
\end{aligned}$$

where the first equality follows from (54), the second from (18) and (53), the third from (57), the fourth from (30), the fifth from (59) and the replacement of indices $i \leftrightarrow r-1-i$, respectively. Finally we use Euler's reflection formula for the gamma function [1, Theorem 1.2.1]:

$$\Gamma(w)\Gamma(1-w) = \frac{\pi}{\sin \pi w}, \tag{60}$$

to establish the desired formula (58) □

With the proof of formula (57) in Proposition 5.4, Theorem 3.7 has been established.

6 Reciprocity and Connection Formula

We shall show that applying Lemma 4.1 to ${}_2K_1(\boldsymbol{a}; z) = {}_2H_1(\boldsymbol{a}; z)$ yields the reciprocity (20) in Definition 2.2. First we observe that under the involution $\lambda \mapsto \check{\lambda}$, the transformation

$$w \mapsto \check{w} := w + c, \qquad c = c(\lambda) := \frac{1-a-b}{r-p-q} \tag{61}$$

also yields an involution, since definition (20) implies $\check{c} := c(\check{\lambda}) = -c(\lambda) = -c$.

Lemma 6.1 *Let $\lambda = (p, q, r; a, b; x)$ be an integral data in domain (21). If λ is a solution to Problem* II *that comes from contiguous relations, with rational function $R(w; \lambda)$ in condition (6), then its reciprocal $\check{\lambda} = (\check{p}, \check{q}, \check{r}; \check{a}, \check{b}; \check{x})$ is also a solution to Problem* II *that comes from contiguous relations, with the corresponding rational function*

$$R(w; \check{\lambda}) = x^r (1-x)^{p+q-r} \cdot \Psi_h(\hat{w}; \lambda) R(\hat{w}; \lambda), \tag{62}$$

where $\Psi_h(w; \lambda)$ is given by formula (51) and $\hat{w}$ is defined by

$$\hat{w} := w - c, \qquad c = c(\lambda) := \frac{1-a-b}{r-p-q}. \tag{63}$$

Proof. Definitions (20) and (61) imply $(r-p-q)w + 1 - a - b = \check{r}\check{w}$ and

$$1 - a - pw = 1 - a - p(\check{w} - c) = \check{p}\check{w} + \check{a},$$

and similarly $1 - b - qw = \check{q}\check{w} + \check{b}$. Substituting these into formula (49) we find $h(w; \lambda) = x^{1-rw}(1-x)^{(r-p-q)w-a-b} f(\check{w}; \check{\lambda})$, or in other words,

$$f(\check{w}; \check{\lambda}) = x^{rw-1}(1-x)^{(p+q-r)w+a+b} h(w; \lambda). \tag{64}$$

Since increasing $\check{w}$ by 1 is equivalent to increasing w by 1, we also have

$$f(\check{w}+1; \check{\lambda}) = x^{r(w+1)-1}(1-x)^{(p+q-r)(w+1)+a+b} h(w+1; \lambda),$$

so that formula (45) in Lemma 4.1 and definition (61) yield

$$\begin{aligned} R(\check{w}; \check{\lambda}) = \frac{f(\check{w}+1; \check{\lambda})}{f(\check{w}; \check{\lambda})} &= x^r (1-x)^{p+q-r} \frac{h(w+1; \lambda)}{h(w; \lambda)} = x^r (1-x)^{p+q-r} \cdot \Psi_h(w; \lambda) R(w; \lambda) \\ &= x^r (1-x)^{p+q-r} \cdot \Psi_h(\check{w} - c; \lambda)\, R(\check{w} - c; \lambda). \end{aligned}$$

Since $\check{w}$ is an indeterminate variable, we can replace $\check{w}$ by w in the above to obtain

$$\begin{aligned} R(w; \check{\lambda}) &= x^r (1-x)^{p+q-r} \cdot \Psi_h(w - c; \lambda) R(w - c; \lambda) \\ &= x^r (1-x)^{p+q-r} \cdot \Psi_h(\hat{w}; \lambda)\, R(\hat{w}; \lambda), \end{aligned}$$

where definition (63) is used in the last equality. □

Using Lemma 6.1 in domains $\mathcal{E}^{*-}$ and $\mathcal{E}^{-*}$ yields the following direct consequence.

Corollary 6.2 *Reciprocity $\lambda \mapsto \check{\lambda}$ in (20) induces a bijection between the set of all solutions $\lambda = (p, q, r; a, b; x) \in \mathcal{E}^{*-}$ with $q \in \mathbb{Z}$ and the set of all solutions $\check{\lambda} = (\check{p}, \check{q}, \check{r}; \check{a}, \check{b}; \check{x}) \in \mathcal{E}^{-*}$ with $\check{p} \in \mathbb{Z}$.*

Proof. In view of definition (20) the reciprocity $\lambda \mapsto \check{\lambda}$ is a bijection $\mathcal{E}^{*-} \to \mathcal{E}^{-*}$ in the data level. By Proposition 3.5 and Lemma 6.1 it induces a bijection in the solution level between the solutions $\lambda \in \mathcal{E}^{*-}$ with $q \in \mathbb{Z}$ and the solutions $\check{\lambda} \in \mathcal{E}^{-*}$ with $\check{p} \in \mathbb{Z}$. □

Although the reciprocity is also a bijection $\mathcal{D}^- \to \mathcal{F}^-$ in the data level, it does not immediately induce a bijection in the integral solution level as in Corollary 6.2. This is because we have not yet known whether every integral solution in $\mathcal{F}^-$ to Problem II comes from contiguous relations, so that the backward reciprocity $\mathcal{F}^- \to \mathcal{D}^-$ is not established yet in the solution level (see Remark 4.2). This issue is postponed until it is settled in Proposition 9.4. In the rest of this section we develop a detailed study of the forward reciprocity $\mathcal{D}^- \to \mathcal{F}^-$ in the integral solution level by using the connection formula for hypergeometric functions.

Lemma 6.3 *If $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ is an* (A)*-solution with GPF* (24)*, then*

$$h(w;\lambda) = \Gamma((r-p-q)w+1-a-b) \cdot \tilde{d}^w \cdot \chi(w) \cdot \frac{\prod_{i=0}^{p-1} \Gamma\left(w+\frac{i+a}{p}\right) \prod_{i=0}^{q-1} \Gamma\left(w+\frac{i+b}{q}\right)}{\prod_{i=1}^{r} \Gamma\left(w+v_i\right)}, \tag{65}$$

where the constant $\tilde{d}$ and the function $\chi(w)$ are given by

$$\tilde{d} = \sqrt{p^p q^q (r-p)^{p-r} (r-q)^{q-r} x^{-r} (1-x)^{r-p-q}}, \tag{66}$$

$$\chi(w) = C_1 \cdot \frac{\sin\pi(pw+a)\sin\pi(qw+b)}{\sin(\pi r w)} + C_2 \cdot \frac{\prod_{i=1}^{r} \sin\pi(w+v_i^*)}{\prod_{i=0}^{r-1} \sin\pi\left(w+\frac{i-1}{r}\right)}, \tag{67}$$

$$C_1 := 2\,(2\pi)^{(r-p-q-1)/2} \cdot C \cdot p^{a-1/2} q^{b-1/2} r^{1/2}, \tag{68}$$

$$C_2 := (2\pi)^{(r-p-q-1)/2} \cdot D \cdot (r-p)^{a+1/2} (r-q)^{b+1/2} r^{-3/2}, \tag{69}$$

with C and D being the constants in formulas (24) *and* (59) *respectively.*

Proof. A connection formula in Erdélyi [6, Chap. II, §2.9, formula (43)]) reads

$$ {}_2H_1(\boldsymbol{a};z) = \frac{\Gamma(\gamma-\alpha-\beta+1)\Gamma(1-\gamma)}{\Gamma(1-\alpha)\Gamma(1-\beta)}\, {}_2F_1(\boldsymbol{a};z) + \frac{\Gamma(\gamma-\alpha-\beta+1)\Gamma(\gamma-1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\, {}_2G_1(\boldsymbol{a};z).$$

Substituting $\boldsymbol{a} = \boldsymbol{\alpha}(w) := (pw+a,\, qw+b;\, rw)$ and $z = x$ into the connection formula and using the definitions of $g(w;\lambda)$ and $h(w;\lambda)$ (just before formulas (48) and (49)), we have

$$h(w;\lambda) = C_f(w)\, f(w;\lambda) + C_g(w)\, g(w;\lambda), \tag{70}$$

where the connection coefficients $C_f(w)$ and $C_g(w)$ are given by

$$C_f(w) = \frac{\Gamma((r-p-q)w+1-a-b)\,\Gamma(1-rw)}{\Gamma(1-a-pw)\,\Gamma(1-b-qw)},$$

$$C_g(w) = \frac{\Gamma((r-p-q)w+1-a-b)\,\Gamma(rw-1)}{\Gamma((r-p)w-a)\Gamma((r-q)w-b)}.$$

The connection coefficient $C_f(w)$ can be written

$$\begin{aligned} C_f(w) &= \Gamma((r-p-q)w+1-a-b) \cdot \frac{\sin\pi(pw+a)\ \sin\pi(qw+b)}{\pi\ \sin(\pi r w)} \cdot \frac{\Gamma(pw+a)\,\Gamma(qw+b)}{\Gamma(rw)} \\ &= \Gamma((r-p-q)w+1-a-b) \cdot \frac{\sin\pi(pw+a)\ \sin\pi(qw+b)}{\pi\ \sin(\pi r w)} \\ &\quad \times (2\pi)^{(r-p-q+1)/2} p^{a-1/2} q^{b-1/2} r^{1/2} \left(\frac{p^p q^q}{r^r}\right)^w \frac{\prod_{i=0}^{p-1} \Gamma\left(w+\frac{i+a}{p}\right) \prod_{i=0}^{q-1} \Gamma\left(w+\frac{i+b}{q}\right)}{\prod_{i=0}^{r-1} \Gamma\left(w+\frac{i}{r}\right)}, \end{aligned}$$

where the first equality follows from the reflection formula (60), while the second equality from Gauss's multiplication formula (12) for the gamma function. The above expression for $C_f(w)$ is multiplied by formula (24) to yield

$$\begin{aligned} C_f(w)\, f(w;\lambda) &= \Gamma((r-p-q)w+1-a-b) \cdot \frac{\sin\pi(pw+a)\ \sin\pi(qw+b)}{\sin(\pi r w)} \\ &\quad \times C_1 \cdot \tilde{d}^w \cdot \frac{\prod_{i=0}^{p-1} \Gamma\left(w+\frac{i+a}{p}\right) \prod_{i=0}^{q-1} \Gamma\left(w+\frac{i+b}{q}\right)}{\prod_{i=0}^{r-1} \Gamma(w+v_i)}, \end{aligned} \tag{71}$$

where one uses definition (68) and relation $d \cdot p^p q^q r^{-r} = \tilde{d}$ that follows from (25) and (66).

In a similar manner the multiplication formula (12) allows us to write

$$C_g(w) = \Gamma((r-p-q)w+1-a-b) \cdot (2\pi)^{(r-p-q-1)/2}(r-p)^{a+1/2}(r-q)^{b+1/2}r^{-3/2}$$
$$\times \left\{ \frac{r^r}{(r-p)^{r-p}(r-q)^{r-q}} \right\}^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(w+\frac{i-1}{r}\right)}{\prod_{i=0}^{r-p-1} \Gamma\left(w+\frac{i-a}{r-p}\right) \prod_{i=0}^{r-q-1} \Gamma\left(w+\frac{i-b}{r-q}\right)},$$

which is multiplied by formula (58) to yield

$$C_g(w)\, g(w;\lambda) = \Gamma((r-p-q)w+1-a-b) \cdot \frac{\prod_{i=1}^{r} \sin \pi(w+v_i)}{\prod_{i=0}^{r-1} \sin \pi\left(w+\frac{i-1}{r}\right)}$$
$$\times C_2 \cdot \tilde{d}^w \cdot \frac{\prod_{i=1}^{r} \Gamma(w+v_i^*)}{\prod_{i=0}^{r-p-1} \Gamma\left(w+\frac{i-a}{r-p}\right) \prod_{i=0}^{r-q-1} \Gamma\left(w+\frac{i-b}{r-q}\right)},$$

where one uses definition (69) and the relation $\delta \cdot r^r(r-p)^{p-r}(r-q)^{q-r} = \tilde{d}$ that follows from (59) and (66). Taking equation (31) into account we have

$$C_g(w)\, g(w;\lambda) = \Gamma((r-p-q)w+1-a-b) \cdot \frac{\prod_{i=1}^{r} \sin \pi(w+v_i)}{\prod_{i=0}^{r-1} \sin \pi\left(w+\frac{i-1}{r}\right)}$$
$$\times C_2 \cdot \tilde{d}^w \cdot \frac{\prod_{i=0}^{p-1} \Gamma\left(w+\frac{i+a}{p}\right) \prod_{i=0}^{q-1} \Gamma\left(w+\frac{i+b}{q}\right)}{\prod_{i=0}^{r-1} \Gamma(w+v_i)}. \tag{72}$$

Substituting (71) and (72) into (70) yields the desired formula (65) with (66) and (67). □

Lemma 6.4 *In Lemma* 6.3 *the function* $\chi(w)$ *in formula* (67) *must be constant so that there exists a real constant* C_3 *such that formula* (65) *becomes*

$$\frac{h(w;\lambda)}{\Gamma((r-p-q)w+1-a-b)} = C_3 \cdot \tilde{d}^w \cdot \frac{\prod_{i=0}^{p-1} \Gamma\left(w+\frac{i+a}{p}\right) \prod_{i=0}^{q-1} \Gamma\left(w+\frac{i+b}{q}\right)}{\prod_{i=1}^{r} \Gamma(w+v_i)}. \tag{73}$$

Proof. The left-hand side of equation (73), which is denoted by $\tilde{h}(w;\lambda)$, is an entire holomorphic function of w, since by formula (49) any pole of $h(w;\lambda)$ must be simple and located where $(r-p-q)w+1-a-b$ becomes a nonpositive integer, so that it is canceled by a zero of $1/\Gamma((r-p-q)w+1-a-b)$. It follows from formula (65) that

$$\chi(w) = \tilde{d}^{-w} \cdot \tilde{h}(w;\lambda) \cdot \frac{\prod_{i=1}^{r} \Gamma(w+v_i)}{\prod_{i=0}^{p-1} \Gamma\left(w+\frac{i+a}{p}\right) \prod_{i=0}^{q-1} \Gamma\left(w+\frac{i+b}{q}\right)},$$

and so $\chi(w)$ is holomorphic in the half-plane $\mathrm{Re}\, w > -\min\{v_i : i = 1, \dots, n\}$. On the other hand formula (67) implies that $\chi(w)$ is a periodic function of period one, since p, q, and r are positive integers with $r-p-q$ even. Thus $\chi(w)$ must be an entire holomorphic and periodic function of period one. In particular $\chi(w)$ is bounded on the horizontal strip $|\mathrm{Im}\, z| \le 1$.

Now notice that $\sin z$ admits a two-sided bound $\frac{1}{4} e^{|\mathrm{Im}\, z|} \le |\sin z| \le e^{|\mathrm{Im}\, z|}$ on the outer region $|\mathrm{Im}\, z| \ge 1$. Applying it to formula (67) yields an estimate:

$$|\chi(w)| \le |C_1| \, \frac{e^{\pi p |\mathrm{Im}\, z|} \cdot e^{\pi q |\mathrm{Im}\, z|}}{\frac{1}{4} e^{\pi r |\mathrm{Im}\, z|}} + |C_2| \, \frac{\prod_{i=1}^{r} e^{\pi |\mathrm{Im}\, z|}}{\prod_{i=0}^{r-1} \frac{1}{4} e^{\pi |\mathrm{Im}\, z|}} = 4|C_1| \, e^{-\pi(r-p-q)|\mathrm{Im}\, z|} + 4^r |C_2|,$$

and hence $|\chi(w)| \le 4|C_1| \, e^{-\pi(r-p-q)} + 4^r |C_2|$ on $|\mathrm{Im}\, z| \ge 1$. Thus the entire function $\chi(w)$ is bounded on $\mathbb{C}_w$. By Liouville's theorem $\chi(w)$ must be constant. $\square$

Given a positive integer k, we put $[k] := \{0, 1, \dots, k-1\}$. By division relation (27) there exist subsets $I_p \subset [p]$, $I_q \subset [q]$, $J_p \subset [r-p]$, $J_q \subset [r-q]$, with $0 \notin I_p$ and $0 \notin I_q$, such that

$$\prod_{i=1}^{r} (w + v_i) = \prod_{i \in I_p} \left(w + \tfrac{i+a}{p} \right) \prod_{i \in I_q} \left(w + \tfrac{i+b}{q} \right) \prod_{j \in J_p} \left(w + \tfrac{j-a}{r-p} \right) \prod_{j \in J_q} \left(w + \tfrac{j-b}{r-q} \right). \tag{74}$$

If we put $\bar{I}_p := [p] \setminus I_p$ and $\bar{I}_q := [q] \setminus I_q$, then equation (73) becomes

$$\frac{h(w; \lambda)}{\Gamma((r-p-q)w + 1 - a - b)} = C_3 \cdot \tilde{d}^{w} \cdot \frac{\prod_{i \in \bar{I}_p} \Gamma\left(w + \frac{i+a}{p} \right) \prod_{i \in \bar{I}_q} \Gamma\left(w + \frac{i+b}{q} \right)}{\prod_{j \in J_p} \Gamma\left(w + \frac{j-a}{r-p} \right) \prod_{j \in J_q} \Gamma\left(w + \frac{j-b}{r-q} \right)}. \tag{75}$$

To exploit formula (75) we need a preliminary lemma. A *multi-set* is a set allowing repeated elements. For multi-sets $S = \{s_1, \dots, s_m\}$ and $T = \{t_1, \dots, t_n\}$, we write $S \succ T$ if $m \le n$ and there exists a re-indexing of $t_1, \dots, t_n$ such that $s_i - t_i \in \mathbb{Z}_{\ge 0}$ for every $i = 1, \dots, m$.

Lemma 6.5 *Let $S = \{s_1, \dots, s_m\}$ and $T = \{t_1, \dots, t_n\}$ be multi-sets of real numbers. If*

$$h(w) = \frac{\Gamma(w + s_1) \cdots \Gamma(w + s_m)}{\Gamma(w + t_1) \cdots \Gamma(w + t_n)}$$

is an entire function of w, then $S \succ T$.

Proof. The proof is by induction on m. When $m = 0$ the assertion is obvious as the numerator of $h(w)$ is 1. Let $m \ge 1$. We may assume $s_1 \le \cdots \le s_m$. An upper factor $\Gamma(w + s_1)$ of $h(w)$ has a pole at $w = -s_1$. In order for $h(w)$ to be holomorphic, a lower factor $\Gamma(w + t_j)$ must have a pole at the same point. After a transposition of t_j and t_1 we may put $j = 1$. Then $-s_1 + t_1 = -r_1 \in \mathbb{Z}_{\le 0}$, that is, $r_1 \in \mathbb{Z}_{\ge 0}$. Since $\Gamma(w + s_1)/\Gamma(w + t_1) = (w + t_1)_{r_1}$, we have

$$h(w) = (w + t_1)_{r_1} \cdot h_1(w), \qquad \text{where} \qquad h_1(w) := \frac{\Gamma(w + s_2) \cdots \Gamma(w + s_m)}{\Gamma(w + t_2) \cdots \Gamma(w + t_n)}.$$

We claim that $h_1(w)$ is entire holomorphic. If $r_1 = 0$ this is obvious since $h(w) = h_1(w)$. Let $r_1 \ge 1$. Any pole of $\Gamma(w + s_2) \cdots \Gamma(w + s_m)$ is separated from all the roots of $(w + t_1)_{r_1}$, because we have $-s_m \le \cdots \le -s_2 \le -s_1$ and the roots of $(w + t_1)_{r_1}$ are located at $-s_1 + 1 < -s_1 + 2 < \cdots < -t_1$ in an increasing order. Thus any pole of $h_1(w)$, if it exists, cannot be canceled by a root of $(w + t_1)_{r_1}$. Accordingly, $h_1(w)$ has no poles, since $h(w)$ has no poles. Now we can apply the induction hypothesis to $h_1(w)$ to conclude the proof. $\square$

case	type	a	b
1	(I, I)	$\dfrac{pi}{r}$	$\dfrac{qj}{r}$
2	(II, II)	$\dfrac{p\{(r-p)i-qj\}}{r(r-p-q)}$	$\dfrac{q\{(r-q)j-pi\}}{r(r-p-q)}$
3	(I, II)	$\dfrac{pi}{r}$	$\dfrac{q(rj-pi)}{r(r-p)}$
4	(II, I)	$\dfrac{p(ri-qj)}{r(r-q)}$	$\dfrac{qj}{r}$

Table 2: Formula for (a,b) in terms of (p,q,r) and $(i,j) \in \mathbb{Z}^2_{\geq 0}$.

Since the left-hand side of equation (75) is entire holomorphic in w, so must be the gamma products on the right. Thus Lemma 6.5 yields

$$\left\{\frac{i+a}{p}\right\}_{i\in\bar{I}_p} \bigcup \left\{\frac{i+b}{q}\right\}_{i\in\bar{I}_q} \succ \left\{\frac{j-a}{r-p}\right\}_{j\in J_p} \bigcup \left\{\frac{j-b}{r-q}\right\}_{j\in J_q}, \qquad (76)$$

where the both sides above are thought of as multi-sets. Note that $0 \in \bar{I}_p$ and $0 \in \bar{I}_q$.

Lemma 6.6 *For any* (A)*-solution* $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ *there exists a pair* (i,j) *of nonnegative integers such that one of the four cases in Table* 2 *comes about. In particular, in either case* a *and* b *must be rational numbers.*

Proof. Since $0 \in \bar{I}_p$, condition (76) implies that either $(\mathrm{I})_p$ or $(\mathrm{II})_p$ below holds:

(I_p) there exists an integer $i \in J_p + (r-p)\,\mathbb{Z}_{\geq 0}$ such that $\dfrac{a}{p} = \dfrac{i-a}{r-p}$,

(II_p) there exists an integer $i \in J_q + (r-q)\,\mathbb{Z}_{\geq 0}$ such that $\dfrac{a}{p} = \dfrac{i-b}{r-q}$.

In a similar manner, since $0 \in \bar{I}_q$, condition (76) implies that either $(\mathrm{I})_q$ or $(\mathrm{II})_q$ below holds:

(I_q) there exists an integer $j \in J_q + (r-q)\,\mathbb{Z}_{\geq 0}$ such that $\dfrac{b}{q} = \dfrac{j-b}{r-q}$,

(II_q) there exists an integer $j \in J_p + (r-p)\,\mathbb{Z}_{\geq 0}$ such that $\dfrac{b}{q} = \dfrac{j-a}{r-p}$.

There are a total of four types $(\mathrm{I}_p,\, \mathrm{I}_q)$, $(\mathrm{II}_p,\, \mathrm{II}_q)$, $(\mathrm{I}_p,\, \mathrm{II}_q)$, $(\mathrm{II}_p,\, \mathrm{I}_q)$, which are exactly the Cases 1–4 of Table 2 respectively, where suffixes $* \in \{p,\, q\}$ of I_* and II_* are omitted. In each case we have a pair of linear equations for (a,b), which can be solved as indicated in Table 2. $\square$

Lemma 6.7 *If* $a > 0$ *then* $0 \notin J_p$*. Similarly if* $b > 0$ *then* $0 \notin J_q$*.*

case	1	2	3	4	5	6	7
type	I I I I	II II II II	I II I II	I I I II	II II II I	I II II I	I I II II

Table 3: Types of matrix (19) up to its column and/or row exchanges.

Proof. We use assertion (1) of Iwasaki [8, Proposition 11.14] with $k = 0$. In the notation there it states that for each $k = 0, \dots, r - p - 1$, one has $(w - w_k^*) | \prod_{i=1}^{r}(w + v_i)$ if and only if

$$(\gamma_k^* + k)_{p+1} \cdot \mathcal{F}_k(\beta_k^*; \gamma_k^*; x) = 0, \tag{77}$$

unless [8, condition (122a)] is satisfied, that is, unless

$$\tilde{\beta}_k^*,\ \tilde{\gamma}_k^* \in \mathbb{Z}, \qquad 0 \le -\tilde{\beta}_k^* \le -\tilde{\gamma}_k^* \le r - p - k - 2. \tag{78}$$

When $k = 0$, definitions in [8, formula (117) and Proposition 11.12] read

$$w_0^* = \frac{a}{r-p}, \qquad \gamma_0^* = r w_0^* = \frac{ra}{r-p}, \qquad \tilde{\gamma}_0^* = 2 - r(w_0^* + 1) = 2 - r - \frac{ra}{r-p},$$

so condition $-\tilde{\gamma}_k^* \le r - p - k - 2$ in (78) is equivalent to $a \le -p(r-p)/r$ (< 0). Thus if $a > 0$ then condition (78) with $k = 0$ is not fulfilled. When $k = 0$, condition (77) becomes $(\gamma_0^*)_{p+1} = 0$, since $\mathcal{F}_0(\beta_0^*; \gamma_0^*; x) = 1$ by [8, definition (116)]. But if $a > 0$ then $\gamma_0^* > 0$ and so $(\gamma_0^*)_{p+1} > 0$, which means that $w - w_0^* = w + \frac{0-a}{r-p}$ is *not* a factor of $\prod_{i=1}^{r}(w + v_i)$. This in turn implies $0 \notin J_p$ by formula (74). The implication $b > 0 \Rightarrow 0 \notin J_q$ is proved in the same way. □

7 Duality Revisited and Finite Cardinality

Consider a solution $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^-$ of type (A) and its dual $\lambda = (p, q, r; a', b'; x) \in \mathcal{D}^-$. Then the associated matrix (19) is said to be of type

$$\begin{pmatrix} J_1 & J_2 \\ J_3 & J_4 \end{pmatrix}, \qquad J_1, J_2, J_3, J_4 \in \{\mathrm{I}, \mathrm{II}\}, \tag{79}$$

if the top row (a, b) of matrix (19) is of type (J_1, J_2), while its bottom row (a', b') is of type (J_3, J_4) in the sense of Table 2. There are a total of seven types up to column and/or row exchanges of the matrix, which are tabulated in Table 3, where parentheses in (79) are omitted.

The essential parts of definition (18) for duality can be rewritten as

$$a + a' = 1 - \frac{2p}{r}, \qquad b + b' = 1 - \frac{2q}{r}. \tag{80}$$

In each case of Table 3 we shall see what kinds of consequences are derived from equations (80). In this section we employ the notation introduced in item (1) of Remark 3.9.

Lemma 7.1 *In cases* 1–6 *of Table* 3*, equations* (80) *give rise to the conditions in Table* 1*, regarding items* (1)–(4) *mentioned in Theorem* 3.8.

Proof. This lemma is proved by case-by-case treatments presented below.
Case 1. By Lemma 6.6 there exists a quadruple of nonnegative integers $(i, j; i', j')$ such that

$$a = \frac{pi}{r}, \qquad b = \frac{qj}{r}, \qquad a' = \frac{pi'}{r}, \qquad b' = \frac{qj'}{r}. \tag{81}$$

Substituting formula (81) into equations (80) yields $i + i' = \frac{r}{p} - 2 \in \mathbb{Z}$ and $j + j' = \frac{r}{q} - 2 \in \mathbb{Z}$, which implies $p|r$ and $q|r$, so that we have

$$i + i' = r_p - 2, \qquad j + j' = r_q - 2, \qquad i, j, i', j' \in \mathbb{Z}_{\geq 0}. \tag{82}$$

Formula (81) then becomes

$$a = \frac{i}{r_p}, \qquad b = \frac{j}{r_q}, \qquad a' = \frac{i'}{r_p}, \qquad b' = \frac{j'}{r_q}. \tag{83}$$

Thus all the conditions in case 1 of Table 1 have been obtained.
Case 2. By Lemma 6.6 there exists a quadruple of nonnegative integers $(i, j; i', j')$ such that

$$\begin{aligned} a &= \frac{p\{(r-p)i - qj\}}{r(r-p-q)}, & b &= \frac{q\{(r-q)j - pi\}}{r(r-p-q)}, \\ a' &= \frac{p\{(r-p)i' - qj'\}}{r(r-p-q)}, & b' &= \frac{q\{(r-q)j' - pi'\}}{r(r-p-q)}. \end{aligned} \tag{84}$$

Substituting formula (84) into equations (80) yields

$$(r-p)(i+i') - q(j+j') = (r-p-q)\left(\tfrac{r}{p} - 2\right), \qquad -p(i+i') + (r-q)(j+j') = (r-p-q)\left(\tfrac{r}{q} - 2\right).$$

They are solved with respect to $i + i'$ and $j + j'$ to obtain $i + i' = (r-p-q)/p \in \mathbb{Z}$ and $j + j' = (r-p-q)/q \in \mathbb{Z}$, which imply $p|(r-p-q)$ and $q|(r-p-q)$, so that we have

$$i + i' = (r-p-q)_p, \qquad j + j' = (r-p-q)_q, \qquad i, j, i', j' \in \mathbb{Z}_{\geq 0}. \tag{85}$$

Formula (84) then becomes

$$a = \frac{(r-p)i - qj}{r(r-p-q)_p}, \quad b = \frac{(r-q)j - pi}{r(r-p-q)_q}, \quad a' = \frac{(r-p)i' - qj'}{r(r-p-q)_p}, \quad b' = \frac{(r-q)j' - pi'}{r(r-p-q)_q}. \tag{86}$$

Thus all the conditions in case 2 of Table 1 have been obtained.
Case 3. By Lemma 6.6 there exists a quadruple of nonnegative integers $(i, j; i', j')$ such that

$$a = \frac{pi}{r}, \qquad b = \frac{q(rj - pi)}{r(r-p)}, \qquad a' = \frac{pi'}{r}, \qquad b' = \frac{q(rj' - pi')}{r(r-p)}. \tag{87}$$

Substituting formula (87) into equations (80) yields $i + i' = \frac{r}{p} - 2 \in \mathbb{Z}$ and $r(j+j') - p(i+i') = (r-p)\left(\frac{r}{q} - 2\right)$, the former of which is put into the latter to yields $j + j' = (r-p-q)/q \in \mathbb{Z}$. Thus we have $p|r$ and $q|(r-p-q)$, and hence

$$i + i' = r_p - 2, \qquad j + j' = (r-p-q)_q, \qquad i, j, i', j' \in \mathbb{Z}_{\geq 0}. \tag{88}$$

Formula (87) then becomes

$$a = \frac{i}{r_p}, \qquad b = \frac{r_p\, j - i}{r_p(r-p)_q}, \qquad a' = \frac{i'}{r_p}, \qquad b' = \frac{r_p\, j' - i'}{r_p(r-p)_q}. \tag{89}$$

Thus all the conditions in case 3 of Table 1 have been obtained.

Case 4. By Lemma 6.6 there exists a quadruple of nonnegative integers $(i,j;i',j')$ such that

$$a = \frac{pi}{r}, \qquad b = \frac{qj}{r}, \qquad a' = \frac{pi'}{r}, \qquad b' = \frac{q(rj'-pi')}{r(r-p)}. \tag{90}$$

Substituting formula (90) into equations (80) yields

$$i + i' = \tfrac{r}{p} - 2 \in \mathbb{Z}, \qquad (r-p)j - pi' + rj' = (r-p)\left(\tfrac{r}{q} - 2\right).$$

The former equation implies $p|r$ and $i + i' = r_p - 2$, while the division of the latter by p makes $(r_p - 1)j - i' + r_p\, j' = (r_p - 1)\left(\frac{r}{q} - 2\right)$, to which $i + i' = r_p - 2$ is added to give $i + (r_p - 1)j + r_p\, j' = r_p(r-p-q)/q \in \mathbb{Z}$. Summing up we have $q|r_p(r-p-q)$ and

$$i + i' = r_p - 2, \qquad i + (r_p - 1)j + r_p\, j' = (r_p(r-p-q))_q, \qquad i,j,i',j' \in \mathbb{Z}_{\geq 0}. \tag{91}$$

Formula (90) then becomes

$$a = \frac{i}{r_p}, \qquad b = \frac{qj}{r}, \qquad a' = \frac{i'}{r_p}, \qquad b' = \frac{r_p\, j' - i'}{(r_p(r-p))_q}. \tag{92}$$

Thus all the conditions in case 4 of Table 1 have been obtained.

Case 5. By Lemma 6.6 there exists a quadruple of nonnegative integers $(i,j;i',j')$ such that

$$a = \frac{p\{(r-p)i - qj\}}{r(r-p-q)}, \qquad b = \frac{q\{(r-q)j - pi\}}{r(r-p-q)}, \qquad a' = \frac{p(ri' - qj')}{r(r-q)}, \qquad b' = \frac{qj'}{r}. \tag{93}$$

Substituting formula (93) into equations (80) yields

$$(r-p)(r-q)i - q(r-q)j + r(r-p-q)i' - q(r-p-q)j' = (r-p-q)(r-q)\left(\tfrac{r}{p} - 2\right), \tag{94a}$$

$$-pi + (r-q)j + (r-p-q)j' = (r-p-q)\left(\tfrac{r}{q} - 2\right). \tag{94b}$$

Calculating (94a) $+\, q \times$ (94b) we have $i + i' = (r-p-q)/p \in \mathbb{Z}$, which implies $p|(r-p-q)$ and so $i + i' = (r-p-q)_p$. Division of (94b) by p makes $-i + (r-q)_p\, j + (r-p-q)_p\, j' = (r-p-q)_p\left(\frac{r}{q} - 2\right)$, to which $i + i' = (r-p-q)_p$ is added to yield $i' + (r-q)_p\, j + (r-p-q)_p\, j' = (r-q)(r-p-q)_p/q \in \mathbb{Z}$. Summing up we have $q|r(r-p-q)_p$ and

$$i + i' = (r-p-q)_p, \qquad i, i' \in \mathbb{Z}_{\geq 0}, \tag{95a}$$

$$i' + (r-q)_p\, j + (r-p-q)_p\, j' = ((r-q)(r-p-q)_p)_q, \qquad i', j, j' \in \mathbb{Z}_{\geq 0}. \tag{95b}$$

Formula (93) then becomes

$$a = \frac{(r-p)i - qj}{r(r-p-q)_p}, \qquad b = \frac{(r-q)_p\, j - i}{(r(r-p-q)_p)_q}, \qquad a' = \frac{ri' - qj'}{r(r-q)_p}, \qquad b' = \frac{qj'}{r}. \tag{96}$$

Thus all the conditions in case 5 of Table 1 have been obtained.

Case 6. By Lemma 6.6 there exists a quadruple of nonnegative integers $(i, j; i', j')$ such that

$$a = \frac{pi}{r}, \qquad b = \frac{q(rj - pi)}{r(r-p)}, \qquad a' = \frac{p(ri' - qj')}{r(r-q)}, \qquad b' = \frac{qj'}{r}. \tag{97}$$

Substituting formula (97) into equations (80) yields

$$(r-q)i - qj' + ri' = (r-q)\left(\tfrac{r}{p} - 2\right), \qquad -pi + (r-p)j' + rj = (r-p)\left(\tfrac{r}{q} - 2\right).$$

They are recast to $(r-p-q)i + (r-p)i' + qj = (r-p)(r-p-q)/p \in \mathbb{Z}$ and $(r-p-q)j' + pi' + (r-q)j = (r-q)(r-p-q)/q \in \mathbb{Z}$. Thus $p|r(r-p-q)$ and $q|r(r-p-q)$, so that

$$(r-p-q)i + (r-p)i' + qj = ((r-p)(r-p-q))_p, \qquad i, i', j \in \mathbb{Z}_{\geq 0}, \tag{98a}$$
$$(r-p-q)j' + pi' + (r-q)j = ((r-q)(r-p-q))_q, \qquad j', i', j \in \mathbb{Z}_{\geq 0}. \tag{98b}$$

Formula (97) then becomes

$$a = \frac{pi}{r}, \qquad b = \frac{rj - pi}{(r(r-p))_q}, \qquad a' = \frac{ri' - qj'}{(r(r-q))_p}, \qquad b' = \frac{qj'}{r}. \tag{99}$$

Thus all the conditions in case 6 of Table 1 have been obtained. □

Lemma 7.2 *Case* 7 *of Table* 3 *cannot occur.*

Proof. By Lemma 6.6 there exists a quadruple of nonnegative integers $(i, j; i', j')$ such that

$$a = \frac{pi}{r}, \qquad b = \frac{qj}{r}, \qquad a' = \frac{p\{(r-p)i' - qj'\}}{r(r-p-q)}, \qquad b' = \frac{q\{(r-q)j' - pi'\}}{r(r-p-q)}.$$

Substituting these into relations (80) yields

$$(r-p-q)i + (r-p)i' - qj' = (r-p-q)\left(\tfrac{r}{p} - 2\right),$$
$$(r-p-q)j + (r-q)j' - pj' = (r-p-q)\left(\tfrac{r}{q} - 2\right),$$

which can readily be converted into

$$(r-q)i + qj + ri' = \tfrac{r}{p} - 2 \in \mathbb{Z}, \tag{100a}$$
$$pi + (r-p)j + rj' = \tfrac{r}{q} - 2 \in \mathbb{Z},, \tag{100b}$$

which imply $p|r$ and $q|r$. Now r must be even, for otherwise r is odd and hence so are p and q by $p|r$ and $q|r$, but then $r - p - q \equiv 1 - 1 - 1 \equiv 1 \mod 2$, that is, $r - p - q$ is odd, which is absurd since it must be even by assertion (2) of Theorem 3.2. If $i' \geq 1$ then (100a) gives an absurd estimate $r = (r-q)\cdot 0 + q \cdot 0 + r \cdot 1 \leq (r-q)i + qj + ri' = r_p - 2 \leq r - 2$. Thus we must have $i' = 0$. In a similar manner equation (100b) forces $j' = 0$. Therefore (100) reduces to

$$(r-q)i + qj = r_p - 2, \tag{101a}$$
$$pi + (r-p)j = r_q - 2,, \tag{101b}$$

If $i \geq 1$ and $j \geq 1$ then (101a) gives an absurd estimate $r = (r-q)\cdot 1 + q \cdot 1 \leq (r-q)i + qj = r_p - 2 \leq r - 2$. Thus we must have either $i = 0$ or $j = 0$. Here we may and shall assume $j = 0$ due to the symmetry of conditions (101a) and (101b). Then (101a) and (101b) yield

$$i = \frac{r_p - 2}{r - q} = \frac{r_q - 2}{p} \in \mathbb{Z}, \tag{102}$$

which in particular implies $p|(r_q - 2)$. Now r_q must be even, for otherwise r_q is odd and hence so is p by $p|(r_q - 2)$, while q is even since $r = qr_q$ is even with r_q odd, but then $r - p - q \equiv 0 - 1 - 0 \equiv 1 \mod 2$, that is, $r - p - q$ is odd, which is again absurd. By the second equality in (102) we have $2q = 4r - 2p - r \cdot r_q = 4r - 2p - 2r \cdot r_{2q}$, that is, $q/p = 2r_p - 1 - r_p \cdot r_{2q} \in \mathbb{Z}$, where r_{2q} makes sense as r_q is even. Thus $p|q$ and so we have integer equations $1 = 2r_p - q_p - r_p \cdot r_{2q} = 4r_{2q} \cdot q_p - q_p - 2(r_{2q})^2 q_p = q_p \cdot \{4r_{2q} - 2(r_{2q})^2 - 1\}$, where $r_p = 2r_{2q} \cdot q_p$ is used in the second equality. Thus $q_p = 1$ and $4r_{2q} - 2(r_{2q})^2 - 1 = 1$, the former of which means $p = q$ while the latter yields $r_{2q}(2 - r_{2q}) = 1$, that is, $r_{2q} = 1$ and so $r = 2q$. Then $r - p - q = r - 2q = 0$, which is absurd since we have $r - p - q > 0$ in $\mathcal{D}^-$. This last contradiction shows that the occurrence of case 7 in Table 3 is impossible. $\square$

Lemma 7.3 *We must have* $-1 < a,\, b,\, a',\, b' < 1$.

Proof. The lemma is proved by a case-by-case check.

Case 1. From condition (82) we have $0 \leq i,\, i' \leq r_p - 2$ and $0 \leq j,\, j' \leq r_q - 2$, so that estimate $0 \leq a,\, a',\, b,\, b' < 1$ follows from representation (83).

Case 2. From condition (85) we have $i \geq 0$ and $j \leq (r-p-q)_q$ as well as $i \leq (r-p-q)_p$ and $j \geq 0$. Thus it follows from representation (86) that

$$\begin{aligned}
1 + a &= \frac{r(r-p-q)_p + (r-p)i - qj}{r(r-p-q)_p} \geq \frac{r(r-p-q)_p - q(r-p-q)_q}{r(r-p-q)_p} \\
&= \frac{r(r-p-q)_p - p(r-p-q)_p}{r(r-p-q)_p} = \frac{r-p}{r} > 0, \\
1 - a &= \frac{r(r-p-q)_p - (r-p)i + qj}{r(r-p-q)_p} \geq \frac{r(r-p-q)_p - (r-p)(r-p-q)_p}{r(r-p-q)_p} = \frac{p}{r} > 0,
\end{aligned}$$

which shows $-1 < a < 1$. In similar manners we have $-1 < b,\, a',\, b' < 1$.

Case 3. From condition (88) we have $0 \leq i,\, i' \leq r_p - 2$, so that estimate $0 \leq a,\, a' < 1$ follows from representation (89). Similarly, from condition (88) we have $j \geq 0$ and $i \leq r_p - 2$ as well as $j \leq (r-p-q)_q$ and $i \geq 0$. Thus it follows from representation (89) that

$$\begin{aligned}
1 + b &= \frac{r_p(r-p)_q + r_p j - i}{r_p(r-p)_q} \geq \frac{r_p(r-p)_q - (r_p - 2)}{r_p(r-p)_q} = \frac{r_p(r-p-q)_q + 2}{r_p(r-p)_q} > 0, \\
1 - b &= \frac{r_p(r-p)_q - r_p j + i}{r_p(r-p)_q} \geq \frac{r_p(r-p)_q - r_p(r-p-q)_q}{r_p(r-p)_q} = \frac{1}{(r-p)_q} > 0,
\end{aligned}$$

which shows $-1 < b < 1$. In a similar manner we have $-1 < b' < 1$.

Case 4. From condition (91) we have $0 \leq i,\, i' \leq r_p - 2$, so that estimate $0 \leq a,\, a' < 1$ follows from representation (92). Similarly, from condition (91) we have $0 \leq (r_p - 1)j \leq (r_p(r-p-q))_q$, that is, $0 \leq qj \leq r(r-p-q)/(r-p)$. Thus representation (92) yields

$$0 \leq b = \frac{qj}{r} \leq \frac{r-p-q}{r-p} < 1.$$

From condition (91) we have $j' \geq 0$ and $i' \leq r_p - 2$ as well as $r_p j' \leq (r_p(r-p-q))_q$ and $i' \geq 0$. Thus it follows from representation (92) that

$$1 + b' = \frac{(r_p(r-p))_q + r_p j' - i'}{(r_p(r-p))_q} \geq \frac{(r_p(r-p))_q - (r_p - 2)}{(r_p(r-p))_q} = \frac{(r_p(r-p-q))_q + 2}{(r_p(r-p))_q} > 0,$$

$$1 - b' = \frac{(r_p(r-p))_q - r_p j' + i'}{(r_p(r-p))_q} \geq \frac{(r_p(r-p))_q - (r_p(r-p-q))_q}{(r_p(r-p))_q} = \frac{q}{r-p} > 0.$$

Therefore we have $0 \leq a,\ a',\ b < 1$ and $-1 < b' < 1$.

Case 5. From condition (95b) we have $(r-q)_p j \leq ((r-q)(r-p-q)_p)_q$, that is, $qj \leq r-p-q = p(r-p-q)_p$. Thus it follows from representation (96) and $i \geq 0$ that

$$1 + a = \frac{r(r-p-q)_p + (r-p)i - qj}{r(r-p-q)_p} \geq \frac{r(r-p-q)_p - p(r-p-q)_p}{r(r-p-q)_p} = \frac{r-p}{r} > 0.$$

From condition (95a) we have $i \leq (r-p-q)_p$. It then follows from (96) and $j \geq 0$ that

$$1 - a = \frac{r(r-p-q)_p - (r-p)i + qj}{r(r-p-q)_p} \geq \frac{r(r-p-q)_p - (r-p)(r-p-q)_p}{r(r-p-q)_p} = \frac{p}{r} > 0,$$

$$1 + b = \frac{(r(r-p-q)_p)_q + (r-q)_p j - i}{(r(r-p-q)_p)_q} \geq \frac{(r(r-p-q)_p)_q - (r-p-q)_p}{(r(r-p-q)_p)_q} = \frac{r-q}{r} > 0.$$

Since $(r-q)_p j \leq ((r-q)(r-p-q)_p)_q$ and $i \geq 0$, representation (96) yields

$$1 - b = \frac{(r(r-p-q)_p)_q - (r-q)_p j + i}{(r(r-p-q)_p)_q - (r-q)_p j + i} \geq \frac{(r(r-p-q)_p)_q - ((r-q)(r-p-q)_p)_q}{(r(r-p-q)_p)_q - (r-q)_p j + i} = \frac{q}{r} > 0.$$

From condition (95b) we have $(r-p-q)_p j' \leq ((r-q)(r-p-q)_p)_q$, that is, $qj' \leq r-q = p(r-q)_p$. Thus it follows from formula (96) and $i' \geq 0$ that

$$1 + a' = \frac{r(r-q)_p + ri' - qj'}{r(r-q)_p} \geq \frac{r(r-q)_p - p(r-q)_p}{r(r-q)_p} = \frac{r-p}{r} > 0.$$

From condition (95a) we have $i' \leq (r-p-q)_p$. Then it follows from (96) and $j' \geq 0$ that

$$1 - a' = \frac{r(r-q)_p - ri' + qj'}{r(r-q)_p} \geq \frac{r(r-q)_p - r(r-p-q)_p}{r(r-q)_p} = \frac{1}{(r-q)_p} > 0.$$

Finally, since $0 \leq qj' \leq r-q$, we have $0 \leq b' = qj'/r \leq (r-q)/r < 1$.

Case 6. From condition (98a) we have $(r-p-q)i \leq ((r-p)(r-p-q))_p$, that is, $pi \leq r-p$. Thus it follows from representation (99) and $i \geq 0$ that $0 \leq a = pi/r \leq (r-p)/r < 1$. Similarly, we have $0 \leq b' < 1$. From condition (98b) we have $(r-p-q)j' \leq ((r-q)(r-p-q))_q$, that is, $qj' \leq r-q$. Thus it follows from representation (99) and $i' \geq 0$ that

$$1 + a' = \frac{(r(r-q))_p + ri' - qj'}{(r(r-q))_p} \geq \frac{(r(r-q))_p - (r-q)}{(r(r-q))_p} = \frac{r-p}{r} > 0.$$

On the other hand, from condition (98a) we have $(r-p)i' \leq ((r-p)(r-p-q))_p$, that is, $pi' \leq r-p-q$. Thus it follows from representation (99) and $j' \geq 0$ that

$$1 - a' = \frac{(r(r-q))_p - ri' + qj'}{(r(r-q))_p} \geq \frac{(r(r-q))_p - ri'}{(r(r-q))_p} = \frac{(r-q) - pi'}{r-q} \geq \frac{p}{r-q} > 0.$$

In a similar manner we have $-1 < b < 1$. $\square$

Lemma 7.4 *We have $0 \le a,\, b < 1$ and there exist inclusions of multi-sets:*

$$\left\{\frac{i+a}{p}\right\}_{i\in\bar{I}_p} \bigcup \left\{\frac{i+b}{q}\right\}_{i\in\bar{I}_q} \subset \left\{\frac{j-a}{r-p}\right\}_{j\in J_p} \bigcup \left\{\frac{j-b}{r-q}\right\}_{j\in J_q}, \tag{103a}$$

$$\left\{\frac{i+a}{p}\right\}_{i\in I_p} \bigcup \left\{\frac{i+b}{q}\right\}_{i\in I_q} \subset \left\{\frac{j-a}{r-p}\right\}_{j\in \bar{J}_p} \bigcup \left\{\frac{j-b}{r-q}\right\}_{j\in \bar{J}_q}, \tag{103b}$$

where $\bar{I}_p := [p] \setminus I_p$, $\bar{I}_q := [q] \setminus I_q$, $\bar{J}_p := [r-p] \setminus J_p$ and $\bar{J}_q := [r-q] \setminus J_q$.

Proof. Recall that $J_p \subset [r-p] = \{0, \dots, r-p-1\}$. By Lemma 7.3 we have $-a < 1$ so that $(j-a)/(r-p) < (r-p-1+1)/(r-p) = 1$ for any $j \in J_p$. If $a \le 0$ then obviously we have $(j-a)/(r-p) \ge 0$ for any $j \in J_p$. If $a > 0$ then we have $J_p \subset \{1, \dots, r-p-1\}$ by Lemma 6.7 and $-a > -1$ by Lemma 7.3, so that $(j-a)/(r-p) > (1-1)/(r-p) = 0$ for any $j \in J_p$. In either case we have $0 \le (j-a)/(r-p) < 1$ for any $j \in J_p$. In a similar manner we have $0 \le (j-b)/(r-q) < 1$ for any $j \in J_q$. So the multi-set on the right-hand side of (76) lies in the interval $[0,\, 1)$. Since $0 \in \bar{I}_p$ and $0 \in \bar{I}_q$, the numbers a/p and b/q belong to the multi-set on the left so that they must be nonnegative by the binary relation (76). Hence we have $a,\, b \ge 0$, which together with Lemma 7.3 yields the estimate $0 \le a,\, b < 1$.

Since $0 \le a < 1$ and $\bar{I}_p \subset \{0, \dots, p-1\}$ we have $0 = (0+0)/p \le (i+a)/p < (p-1+1)/p = 1$ for any $i \in \bar{I}_p$. In a similar manner we have $0 \le (i+b)/q < 1$ for any $i \in \bar{I}_q$. Thus the multi-set on the left-hand side of (76) also lies in the interval $[0,\, 1)$, as does the multi-set on the right. Therefore binary relation (76) mus be the inclusion (103a).

To prove inclusion (103b), we use the dual version of inclusion (103a):

$$\left\{\frac{i'+a'}{p}\right\}_{i'\in\bar{I}'_p} \bigcup \left\{\frac{i'+b'}{q}\right\}_{i'\in\bar{I}'_q} \subset \left\{\frac{j'-a'}{r-p}\right\}_{j'\in J'_p} \bigcup \left\{\frac{j'-b'}{r-q}\right\}_{j'\in J'_q}, \tag{104}$$

where I'_p, I'_q, J'_p, J'_q are the dual counterparts of I_p, I_q, J_p, J_q, with $\bar{I}'_p := [p] \setminus I'_p$ and $\bar{I}'_q := [r-q] \setminus I'_q$. Dividing equation (31) by (74), we have

$$\prod_{i=1}^{r}(w + v_i^*) = \prod_{i\in\bar{I}_p}\left(w + \tfrac{i+a}{p}\right)\prod_{i\in\bar{I}_q}\left(w + \tfrac{i+b}{q}\right)\prod_{j\in\bar{J}_p}\left(w + \tfrac{j-a}{r-p}\right)\prod_{j\in\bar{J}_q}\left(w + \tfrac{j-b}{r-q}\right).$$

Taking the reflection $w \mapsto w'$ as in (53) and using definitions (18) and (30) there yield

$$\begin{aligned}\prod_{i=1}^{r}(w + v'_i) = {} & \prod_{i\in\bar{I}_p}\left(w + \tfrac{(p-1-i)+a'}{p}\right)\prod_{i\in\bar{I}_q}\left(w + \tfrac{(q-1-i)+b'}{q}\right) \\ & \times \prod_{j\in\bar{J}_p}\left(w + \tfrac{(r-p-1-j)-a'}{r-p}\right)\prod_{j\in\bar{J}_q}\left(w + \tfrac{(r-q-1-j)-b'}{r-q}\right),\end{aligned}$$

which together with the definitions of I'_s, $\bar{I}'_s$ and J'_s, $s \in \{p,\, q\}$, implies

$$\begin{aligned} I'_s &= \{i' = s-1-i \,:\, i \in \bar{I}_s\}, \qquad \bar{I}'_s = \{i' = s-1-i \,:\, i \in I_s\}, \\ J'_s &= \{j' = r-s-1-j \,:\, j \in \bar{J}_s\} \qquad\qquad (s \in \{p,\, q\}). \end{aligned} \tag{105}$$

Under the correspondences $i' \leftrightarrow i$ and $j' \leftrightarrow j$ in (105), it follows from definition (18) that

$$\frac{i'+c'}{s} = 1 - \frac{2}{r} - \frac{i+c}{s}, \qquad \frac{j'-c'}{r-s} = 1 - \frac{2}{r} - \frac{j-c}{r-s}, \qquad ((s,c) = (p,a),\, (q,b)).$$

Thus inclusion (104) and relation (105) lead to inclusion (103b). $\square$

Proposition 7.5 *For any* (A)*-solution* $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ *and its dual* (A)*-solution* $\lambda' = (p,q,r;a',b';x) \in \mathcal{D}^-$ *the pair* (λ,λ') *must be in one of the six cases in Table* 1 *and furthermore we must have*

$$a, b,\ v_1,\dots,v_r \in \mathbb{Q}; \qquad 0 \le a,\ b < 1, \qquad 0 \le v_1,\dots,v_r < 1, \tag{106}$$

along with the same condition for a', b' *and* $v_1',\dots,v_r'$.

Proof. The first assertion concerning Table 1 is an immediate consequence of Lemmas 7.1 and 7.2. To prove the second assertion (106), notice that the rationality of a and b is already proved in Lemma 6.6, while $v_1,\dots,v_r \in \mathbb{Q}$ follows from $a, b \in \mathbb{Q}$ and formula (74). The estimate $0 \le a,\ b < 1$ is already proved in Lemma 7.4, while $0 \le v_1,\dots,v_r < 1$ follows from a combination of $0 \le a,\ b < 1$, formula (74) and Lemma 6.7. Thus we have condition (106). $\square$

Theorem 3.8 is the same as Proposition 7.5, while Theorem 3.11 is contained in the following.

Proposition 7.6 *For any* (A)*-solution* $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ *the division relation* (34) *holds true along with yet another division relation*

$$\prod_{i=0}^{p-1}\left(w+\tfrac{i+a}{p}\right)\prod_{i=0}^{q-1}\left(w+\tfrac{i+b}{q}\right) \ \Big|\ \prod_{j=0}^{r-p-1}\left(w+\tfrac{j-a}{r-p}\right)\prod_{j=0}^{r-q-1}\left(w+\tfrac{j-b}{r-q}\right) \qquad \text{in} \quad \mathbb{Q}[w], \tag{107}$$

and hence one can arrange $v_1,\dots,v_r$ *so that equation* (35) *holds. With this convention the reciprocal* $\check{\lambda}$ *of* λ *becomes a solution to Problem* I *in* $\mathcal{F}^-$ *with gamma product formula*

$$f(w;\check{\lambda}) = \check{C}\cdot \check{d}^{w}\cdot \frac{\prod_{i=0}^{r-p-q-1}\Gamma\left(w+\frac{i}{r-p-q}\right)}{\prod_{i=1}^{r-p-q}\Gamma\left(w+\check{v}_i\right)}, \tag{108}$$

where $\check{C}$ *is a positive constant,* $\check{d}$ *and* $\check{v}_1,\dots,\check{v}_{r-p-q}$ *are given by formulas* (37) *and* (38).

Proof. The inclusions of multi-sets (103a) and (103b) are equivalent to the division relations

$$\prod_{i\in \bar{I}_p}\left(w+\tfrac{i+a}{p}\right)\prod_{i\in \bar{I}_q}\left(w+\tfrac{i+b}{q}\right) \ \Big|\ \prod_{j\in J_p}\left(w+\tfrac{j-a}{r-p}\right)\prod_{j\in J_q}\left(w+\tfrac{j-b}{r-q}\right), \tag{109a}$$

$$\prod_{i\in I_p}\left(w+\tfrac{i+a}{p}\right)\prod_{i\in I_q}\left(w+\tfrac{i+b}{q}\right) \ \Big|\ \prod_{j\in \bar{J}_p}\left(w+\tfrac{j-a}{r-p}\right)\prod_{j\in \bar{J}_q}\left(w+\tfrac{j-b}{r-q}\right), \tag{109b}$$

in $\mathbb{C}[w]$, respectively. Then division relation (34) follows from formula (74) and relation (109a), while division relation (107) is derived by multiplying relations (109a) and (109b) together.

Now that division relation (34) is proved, the convention (35) recasts formula (73) to

$$h(w;\lambda) = C_3\cdot \tilde{d}^{w}\cdot \frac{\Gamma((r-p-q)w+1-a-b)}{\prod_{i=1}^{r-p-q}\Gamma\left(w+v_i\right)}. \tag{110}$$

Replacing w by $\hat{w}$ in formula (64), where $\hat{w}$ is defined by formula (63), we have

$$\begin{aligned}
f(w;\check{\lambda}) &= x^{r\check{w}-1}(1-x)^{(p+q-r)\check{w}+a+b}\, h(\hat{w};\lambda) \\
&= C_3\cdot x^{r\check{w}-1}(1-x)^{(p+q-r)\check{w}+a+b}\, \tilde{d}^{\hat{w}}\cdot \frac{\Gamma((r-p-q)\hat{w}+1-a-b)}{\prod_{i=1}^{r-p-q}\Gamma\left(\hat{w}+v_i\right)} \\
&= C_4\cdot x^{rw}(1-x)^{(p+q-r)w}\, \tilde{d}^{w}\cdot \frac{\Gamma\left((r-p-q)w\right)}{\prod_{i=1}^{r-p-q}\Gamma\left(w+\check{v}_i\right)},
\end{aligned}$$

where (110) is used in the second equality, definitions (38) and (63) are used in the third equality and C_4 is a real constant. Applying the multiplication formula (12) to $\Gamma((r-p-q)w)$ yields

$$f(w;\check{\lambda}) = \check{C} \cdot \{x^r(1-x)^{p+q-r}\,\tilde{d}\,(r-p-q)^{r-p-q}\}^w \cdot \frac{\prod_{i=0}^{r-p-q-1} \Gamma\left(w + \frac{i}{r-p-q}\right)}{\prod_{i=1}^{r-p-q} \Gamma(w+\check{v}_i)},$$

where $\check{C}$ is a real constant. This gives GPF (108), since $x^r(1-x)^{p+q-r}\,\tilde{d}\,(r-p-q)^{r-p-q} = \check{d}$ by definition (37). To see $\check{C} > 0$, we look at formula (108) for a large positive value of w. The right-hand side of it without constant factor $\check{C}$ is positive, while the left-hand side $f(w;\check{\lambda})$ is also positive since $\check{\lambda} \in \mathcal{F}^-$. Here we used the fact that if $\mu \in \mathcal{F}^-$ then $f(w;\mu) > 0$ for every large $w > 0$, which will be shown in the proof of Lemma 9.3; see claim (122). Thus $\check{C} > 0$. □

Remark 7.7 In the situation of Proposition 7.6 we have $\check{a}$, $\check{b} \in \mathbb{Q}$, $\check{x}$ algebraic, and

$$\check{v}_1 + \cdots + \check{v}_{r-p-q} = \frac{r-p-q-1}{2} = \frac{\check{r}-1}{2}, \tag{111}$$

$$\check{v}_1, \dots, \check{v}_{r-p-q} \in \mathbb{Q}, \qquad -c \le \check{v}_1, \dots, \check{v}_{r-p-q} < 1-c, \qquad c := c(\lambda) = \frac{1-a-b}{r-p-q}. \tag{112}$$

Indeed, since a and b are rational by Lemma 6.6 while x is algebraic by assertion (1) of Theorem 3.3, so are $\check{a}$, $\check{b}$ and $\check{x}$ via the definition of reciprocity (20). Moreover summation (111) comes from (26) together with convention (35) and definition (38). while condition (112) follows from (106) via definition (38). These remarks will play an important part in §9.2.

8 Division Relations

In Lemma 7.1 we derived some arithmetical constraints on $\boldsymbol{p} = (p,q;r)$ for (A)-solutions in $\mathcal{D}^-$. In this section we exploit division relation (107) to amplify this kind of study. In what follows let $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ be an arbitrary solution of type (A).

Lemma 8.1 *If $p \nmid r$ and $p \nmid (r-p-q)$, then $p | 2r$ and $p | 2(r-p-q)$ with $p \ge 4$.*

Proof. If $p = 1$ we obviously have $p|r$. If $p = 2$ we have $p|(r-p-q)$ since $r-p-q$ must be even. Hereafter we suppose $p \ge 3$. Since $\prod_{i=0}^{p-1}(w + (i+a)/p)$ divides the right-hand side of (107), there exists a mapping $\phi : [p] \to [r-p] \sqcup [r-q]$ such that for each $i \in [p]$,

$$\frac{i+a}{p} = \begin{cases} \dfrac{\phi(i)-a}{r-p} & \text{if} \quad \phi(i) \in [r-p] \quad (i \text{ is homogeneous, or of type I}), \\[2ex] \dfrac{\phi(i)-b}{r-q} & \text{if} \quad \phi(i) \in [r-q] \quad (i \text{ is heterogeneous, or of type II}). \end{cases} \tag{113}$$

Then a sequence $\tau = (\tau_0, \tau_1, \dots, \tau_{p-1})$ of symbols I and II is defined by assigning $\tau_i = \mathrm{I}$ or $\tau_i = \mathrm{II}$ if i is homogeneous or heterogeneous respectively. There is a dichotomy:

(1) There exists at least one index $i \in [p-1]$ such that $\tau_i = \tau_{i+1}$.

(2) The sequence τ is interlacing, that is, $\tau_i \neq \tau_{i+1}$ for any index $i \in [p-1]$.

We begin with case (1). In this case we have either $\tau_i = \tau_{i+1} = \mathrm{I}$ or $\tau_i = \tau_{i+1} = \mathrm{II}$. In the former subcase, it follows from formula (113) that

$$\frac{i+a}{p} = \frac{\phi(i)-a}{r-p} \quad \text{and} \quad \frac{(i+1)+a}{p} = \frac{\phi(i+1)-a}{r-p}, \quad \text{so that} \quad \frac{1}{p} = \frac{\phi(i+1)-\phi(i)}{r-p},$$

that is, $r-p = p\{\phi(i+1)-\phi(i)\}$, which implies $p|r$. In the latter subcase, similarly we have

$$\frac{i+a}{p} = \frac{\phi(i)-b}{r-q} \quad \text{and} \quad \frac{(i+1)+a}{p} = \frac{\phi(i+1)-b}{r-q}, \quad \text{so that} \quad \frac{1}{p} = \frac{\phi(i+1)-\phi(i)}{r-q},$$

that is, $r-q = p\{\phi(i+1)-\phi(i)\}$, which implies $p|(r-p-q)$.

We proceed to case (2) with $p=3$. In this case we have either $\tau = (\mathrm{I}, \mathrm{II}, \mathrm{I})$ or $\tau = (\mathrm{II}, \mathrm{I}, \mathrm{II})$. In the former subcase, it follows from (113) and $\tau_0 = \tau_2 = \mathrm{I}$ that

$$\frac{a}{p} = \frac{\phi(0)-a}{r-p} \quad \text{and} \quad \frac{2+a}{p} = \frac{\phi(2)-a}{r-p}, \quad \text{so that} \quad \frac{2}{p} = \frac{\phi(2)-\phi(0)}{r-p},$$

that is, $2(r-p) = p\{\phi(2)-\phi(0)\}$, which implies $p|2r$, but since $p=3$ is odd, we must have $p|r$. In the latter subcase, a similar reasoning with $\tau_0 = \tau_2 = \mathrm{II}$ leads to $2(r-q) = p\{\phi(2)-\phi(0)\}$, which implies $p|2(r-p-q)$, but since $p=3$ is odd, we must have $p|(r-p-q)$.

Finally we consider case (2) with $p \ge 4$. In this case we have either $(\tau_0, \tau_1, \tau_2, \tau_3) = (\mathrm{I}, \mathrm{II}, \mathrm{I}, \mathrm{II})$ or $(\tau_0, \tau_1, \tau_2, \tau_3) = (\mathrm{II}, \mathrm{I}, \mathrm{II}, \mathrm{I})$. In the former subcase, it follows from (113) that

$$\frac{a}{p} = \frac{\phi(0)-a}{r-p} \quad \text{and} \quad \frac{2+a}{p} = \frac{\phi(2)-a}{r-p}, \quad \text{so that} \quad \frac{2}{p} = \frac{\phi(2)-\phi(0)}{r-p},$$

$$\frac{1+a}{p} = \frac{\phi(1)-b}{r-q} \quad \text{and} \quad \frac{3+a}{p} = \frac{\phi(3)-b}{r-q}, \quad \text{so that} \quad \frac{2}{p} = \frac{\phi(3)-\phi(1)}{r-q},$$

that is, $2(r-p) = p\{\phi(2)-\phi(0)\}$ and $2(r-q) = p\{\phi(3)-\phi(1)\}$, which imply $p|2r$ and $p|2(r-p-q)$. In the latter subcase, a similar reasoning leads to $2(r-p) = p\{\phi(3)-\phi(1)\}$ and $2(r-q) = p\{\phi(2)-\phi(0)\}$, which again imply $p|2r$ and $p|2(r-p-q)$. Thus if $p \not| r$ and $p \not| (r-p-q)$, then we must be in case (2) with $p \ge 4$, which forces $p|2r$ and $p|2(r-p-q)$. $\square$

Lemma 8.2 *If $p \not| r$ and $p \not| (r-p-q)$, then*

(1) *there exist positive integers s, t and k such that*

$$p = 2^{k+1}s; \qquad p|q; \qquad r = 2^k t; \qquad s,\, t : \textit{odd}; \qquad s|t, \tag{114}$$

(2) *we must also have $q \not| r$ and $q \not| (r-p-q)$.*

Proof. We use Lemma 8.1, upon writing $p = 2^i s$, $r = 2^k t$, $r-q = 2^j u$ with $i, j, k \in \mathbb{Z}_{\ge 0}$ and odd $s, t, u \in \mathbb{N}$. Division relation $p|2r$ implies $i \le k+1$ and $s|t$, while $p \not| r$ yields $i \ge k+1$ and hence $i = k+1$. In a similar manner the division relation $p|2(r-q)$ implies $i \le j+1$ and $s|u$, while $p \not| (r-q)$ yields $i \ge j+1$ and hence $i = j+1$. In summary we have

$$p = 2^{k+1}s; \quad r = 2^k t; \quad r-q = 2^k u; \quad k \in \mathbb{Z}_{\ge 0}; \quad s,\, t,\, u : \text{odd}; \quad s|t, \quad s|u.$$

Since t and u are odd, we can write $t = 2t' + 1$ and $u = 2u' + 1$ with $t', u' \in \mathbb{Z}_{\geq 0}$, so that $s|t$ and $s|u$ imply $s|(t-u) = s|2(t'-u')$, which in turn yields $s|(t'-u')$ since s is also odd. It then implies $p|q$ because $p = 2^{k+1}\, s$ and $q = r - (r-q) = 2^k(t-u) = 2^{k+1}(t'-u')$. Note that p is even by $p = 2^{k+1}\, s$ with $k \geq 0$, so q is even too by $p|q$ and r is also even, since $r-p-q$ is even. Thus we have $k \geq 1$ and all the conditions in (114) have been proved.

Next we show that $q \not| r$ and $q \not|(r-p-q)$ by contradiction. Indeed, if $q|r$ then the division relation $p|q$ in condition (114) yields $p|r$ contrary to the assumption $p \not| r$, while if $q|(r-p-q)$ then $p|q$ gives $p|(r-p-q)$ contrary to the assumption $p \not|(r-p-q)$. □

Note that Lemmas 8.1 and 8.2 remain true if the roles of p and q are exchanged.

Proposition 8.3 *For any* (A)*-solution* $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ *the integer vector* $\boldsymbol{p} = (p,q;r) \in D_{\mathrm{A}}^-$ *must satisfy division relations* (28)*.*

Proof. To prove the lemma by contradiction, suppose the contrary that

$$(\, p \not| r \quad \text{and} \quad p \not|(r-p-q)\,) \quad \text{or} \quad (\, q \not| r \quad \text{and} \quad q \not|(r-p-q)\,).$$

By symmetry we may take the former condition in the "or" sentence above, but the latter condition also follows from assertion (2) of Lemma 8.2, so that we are led to the "and" sentence:

$$(\, p \not| r \quad \text{and} \quad p \not|(r-p-q)\,) \quad \text{and} \quad (\, q \not| r \quad \text{and} \quad q \not|(r-p-q)\,).$$

By a part of condition (114) we have $p|q$ and likewise $q|p$ upon exchanging the role of p and q. Hence $p = q$ and condition (114) becomes

$$p = q = 2^{k+1}\, s; \qquad r = 2^k\, t; \qquad k \in \mathbb{N}, \quad s,\, t : \text{odd}; \qquad s|t, \tag{115}$$

A look at Table 1 shows that if $p = q$ then we must have $p|r$ in cases (1)–(5), while currently we have the contrary $p \not| r$. Thus we must be in case 6 of Table 1, so that $\mathbb{Z}$-linear equation (98a) must be satisfied. It follows from formula (115) that $r-p-q = 2^k s(t_s - 4)$, $r - p = 2^k s(t_s - 2)$, $q = 2^{k+1}s$ and $((r-p)(r-p-q))_p = 2^{k-1}s(t_s-2)(t_s-4)$, so equation (98a) is equivalent to

$$(t_s - 4)i + (t_s - 2)i' + 2j = \frac{1}{2}(t_s-2)(t_s-4), \qquad i, i', j \in \mathbb{Z}_{\geq 0},$$

where we used the notation introduced in item (1) of Remark 3.9. Notice that the left-hand side above is an integer, while the right-hand side is a half-integer (not an integer), because s and t are odd integers with $s|t$ so $t_s = t/s$ is also an odd integer. This contradiction shows that our starting assumption is false and condition (28) must be true. □

Theorem 3.6 is now established because it is the same as Proposition 8.3. By definition (22), if $\boldsymbol{p} = (p,q;r) \in D_{\mathrm{A}}^-$ then $\check{r} := r - p - q \in 2\mathbb{N}$. Conversely we have the following.

Lemma 8.4 *Given any* $\check{r} \in 2\mathbb{N}$*, there are only a finite number of triples* $\boldsymbol{p} = (p,q;r) \in D_{\mathrm{A}}^-$ *that satisfy* $r - p - q = \check{r}$ *and the division relations* (28)*; any such* $\boldsymbol{p}$ *must be bounded by*

$$1 \leq p,\, q \leq 3\check{r}, \qquad 2 \leq p + q \leq 5\check{r}, \qquad 4 \leq r \leq 6\check{r}. \tag{116}$$

Proof. Since $p,\, q \in \mathbb{N}$ and $r \in 2\mathbb{N}$, it is evident that $p,\, q \ge 1$ and $r = \check{r} + p + q \ge 4$. Division relation (28) is equivalent to the condition that ($p|\check{r}$ or $p|(\check{r}+q)$) and ($q|\check{r}$ or $q|(\check{r}+p)$), which is divided into four cases: (i) $p|\check{r}$ and $q|\check{r}$; (ii) $p|\check{r}$ and $q|(\check{r}+p)$; (iii) $p|(\check{r}+q)$ and $q|\check{r}$; (iv) $p|(\check{r}+q)$ and $q|(\check{r}+p)$. In case (i) we have $p,\, q \le \check{r}$ and $r = \check{r}+p+q \le 3\check{r}$. In case (ii) we have $p \le \check{r}$, $q \le \check{r}+p \le 2\check{r}$ and $r = \check{r}+p+q \le 4\check{r}$. Case (iii) is similar to case (ii). In case (iv) there exist $i,\, j \in \mathbb{N}$ such that $\check{r}+q = ip$ and $\check{r}+p = jq$. Note that $ij \ge 2$, for otherwise $i = j = 1$ would imply $\check{r} = p - q = -\check{r} = 0$, a contradiction to $\check{r} \ge 4$. Thus the two equations above for (p, q) are uniquely settled as $p = l\,\check{r}$, $q = m\,\check{r}$ and hence $r = \check{r}+p+q = n\,\check{r}$, where

$$l = \frac{j+1}{ij-1}, \qquad m = \frac{i+1}{ij-1}, \qquad n = 1+l+m = \frac{(i+1)(j+1)}{ij-1} \qquad (i, j \in \mathbb{N};\ ij \ge 2).$$

To estimate these numbers we may assume $i \ge j$ and so $i \ge 2$ and $j \ge 1$ by symmetry. Then

$$l \le \frac{j+1}{2j-1} = \frac{1}{2} + \frac{3}{2(2j-1)} \le \frac{1}{2} + \frac{3}{2} = 2, \qquad m \le \frac{i+1}{i-1} = 1 + \frac{2}{i-1} \le 1+2 = 3,$$

and $n = 1+l+m \le 1+2+3 = 6$. This establishes the bound (116). $\square$

For example, those $\boldsymbol{p} = (p, q; r) \in D_{\mathrm{A}}^{-}$ with $\check{r} = 2$ and $p \ge q$ are exactly $\boldsymbol{p} = (1,1;4)$, $(2,1;5)$, $(2,2;6)$, $(3,1;6)$, $(4,2;8)$, among which only $(2,1;5)$ leads to no solutions (see [8, Table 2]).

9 The South-West Domain

We discuss Problem 1.1 (regarding the relationship between Problems I and II) and Problem 1.4 (regarding the method of contiguous relations) on the south-west domain $\mathcal{F}^{-}$. The corresponding work on the cross-shaped domain $\mathcal{D} \cup \mathcal{I} \cup \mathcal{E}$ was carried out in [8], where Problem 1.1 was treated first (§5) and Problem 1.4 was discussed later (§11.1). This logical order will be reversed on the domain $\mathcal{F}^{-}$, that is, we first tackle Problem 1.4 (as well as Problem 1.5) and then proceed to Problem 1.1. The main tool for this passage is the use of reciprocity.

9.1 Coming from Contiguous Relations

In solving Problem 1.4 it is important to think of the linear independence of $f(w;\lambda)$ and $\tilde{f}(w;\lambda)$ over the rational function field $\mathbb{C}(w)$, where $\tilde{f}(w;\lambda)$ is defined right after formula (9). This issue was already discussed in [8, §11.1] on the domain $\mathcal{D} \cup \mathcal{I} \cup \mathcal{E}$, where the equivalence of Problems I and II (Theorem 3.1) made it more tractable. Without such an advantage, our discussion on $\mathcal{F}^{-}$ should be more elaborate and require some function-theoretic preliminaries.

Lemma 9.1 *If $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_k$ are real numbers with $\alpha_1 \ge \cdots \ge \alpha_k > 0$, then for any integer $m \ge \beta_1$ there exists a positive number $\rho_m \in I_m$ such that*

$$\prod_{j=1}^{k} \frac{1}{|\sin \pi(\alpha_j w + \beta_j)|} \le C \qquad \text{on the circle} \quad |w| = \rho_m, \tag{117}$$

where C is a positive constant independent of m and I_m is an open interval defined by

$$I_m := \left(\frac{m-\beta_1}{\alpha_1},\ \frac{m+1-\beta_1}{\alpha_1} \right) \qquad (m \in \mathbb{Z}).$$

Proof. Note that I_m is an open interval of width $1/\alpha_1$, whose endpoints are a pair of consecutive zeros of $\sin \pi(\alpha_1 w+\beta_1)$. For each $j = 2, \dots, k$, all the zeros of $\sin \pi(\alpha_j w+\beta_j)$ form an arithmetic progression $w_{j,m} := \frac{m-\beta_j}{\alpha_j}$ ($m \in \mathbb{Z}$) of common difference $1/\alpha_j \ge 1/\alpha_1$, so that I_m can contain at most one zero of $\sin \pi(\alpha_j w+\beta_j)$. Thus I_m can contain at most $k-1$ zeros of $\prod_{j=2}^{k} \sin \pi(\alpha_j w+\beta_j)$, which partition I_m into at most k subintervals. Among them let J_m be a subinterval of the largest width. By pigeon hole principle the width of J_m cannot be smaller than $\frac{1}{k\alpha_1}$, so that the midpoint ρ_m of J_m is at least $\delta := \frac{1}{2k\alpha_1}$ distant from all the zeros of $\prod_{j=1}^{k} \sin \pi(\alpha_j w + \beta_j)$. For $j = 1, \dots, k$ and $m \in \mathbb{Z}$ let $D_{j,m}$ be the open disk of radius δ with center at $w_{j,m}$. Then it is not hard to see that there exists a positive constant C_j such that

$$\frac{1}{|\sin \pi(\alpha_j w + \beta_j)|} \le C_j \qquad \text{for all} \quad w \in \mathbb{C} \setminus D_j, \quad \text{with} \quad D_j := \bigcup_{m \in \mathbb{Z}} D_{j,m}.$$

Accordingly, if we set $C := C_1 \cdots C_k$ then we have

$$\prod_{j=1}^{k} \frac{1}{|\sin \pi(\alpha_j w + \beta_j)|} \le C \qquad \text{for all} \quad w \in K := \mathbb{C} \setminus \bigcup_{j=1}^{k} D_j. \tag{118}$$

Notice that $\rho_m \in K$ for any $m \in \mathbb{Z}$. If $m \ge \beta_1$ then $\rho_m > 0$ and the circle $|w| = \rho_m$ is contained in K, hence estimate (118) leads to estimate (117). $\square$

Lemma 9.2 *If $\lambda = (p, q, r; a, b; x)$ lies in domain (17) then there exists an infinite sequence of positive numbers $\{\rho_m\}_{m \ge m_0}$ such that for any integer $m \ge m_0$,*

$$\sigma m + \tau < \rho_m < \sigma(m+1) + \tau, \qquad |f(w; \lambda)| \le M_1 e^{c_1 \rho_m} \quad \text{on the circle} \quad |w| = \rho_m, \tag{119}$$

where M_1, c_1, $\sigma > 0$ and $\tau \in \mathbb{R}$ are independent of m and m_0 is an integer with $\sigma m_0 + \tau \ge 0$.

Proof. The hypergeometric function admits Euler's contour integral representation

$${}_2F_1(\alpha, \beta; \gamma; z) = -\frac{e^{-\pi i \gamma}}{4 \sin \pi\beta \cdot \sin \pi(\gamma - \beta) \cdot B(\beta, \gamma - \beta)} \int_{\wp} t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt,$$

along a Pochhammer loop $\wp$ around $t = 0$ and $t = 1$. Notice that

$$\begin{aligned}
\frac{1}{B(\beta, \gamma - \beta)} &= (1-\gamma) \cdot \frac{\sin \pi\beta \cdot \sin \pi(\gamma - \beta)}{\pi \sin \pi\gamma} \cdot B(1-b, 1-(\gamma-\beta)) \\
&= -\frac{(1-\gamma) e^{\pi i \gamma}}{4\pi \cdot \sin \pi\gamma} \int_{\wp} t^{-\beta}(1-t)^{-(\gamma-\beta)} dt,
\end{aligned}$$

where the first equality follows from the reflection formula for the beta function:

$$B(\alpha, \beta) B(1-\alpha, 1-\beta) = \frac{\pi \sin \pi(\alpha+\beta)}{(1-\alpha-\beta) \cdot \sin \pi\alpha \cdot \sin \pi\beta},$$

while the second equality stems from Euler's integral representation of the beta function along the Pochhammer loop $\wp$. Putting these together we have

$$\begin{aligned}
{}_2F_1(\alpha, \beta; \gamma; z) &= \frac{1-\gamma}{16\pi \cdot \sin \pi\beta \cdot \sin \pi\gamma \cdot \sin \pi(\gamma-\beta)} \\
&\quad \times \left(\int_{\wp} t^{-\beta}(1-t)^{-(\gamma-\beta)} dt \right) \left(\int_{\wp} t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt \right),
\end{aligned}$$

which evaluated at $(\alpha, \beta; \gamma; z) = (pw + a, qw + b; rw; x)$ yields $f(w; \lambda) = \psi_1(w)\psi_2(w)$ with

$$\psi_1(w) = \frac{1 - rw}{16\pi \cdot \sin \pi(qw + b) \cdot \sin(\pi rw) \cdot \sin \pi((r - q)w - b)},$$

$$\psi_2(w) = \left(\int_{\wp} t^{-qw-b}(1 - t)^{-(r-q)w+b} dt\right) \left(\int_{\wp} t^{qw+b-1}(1 - t)^{(r-q)w-b-1}(1 - xt)^{-pw-a} dt\right).$$

We can apply Lemma 9.1 to the first factor $\psi_1(w)$. For some constants $\sigma > 0$ and $\tau \in \mathbb{R}$ there exists an infinite sequence $\rho_m \in (\sigma m + \tau,\ \sigma(m + 1) + \tau)$ such that $|\psi_1(w)| = O(\rho_m)$ on the circle $|w| = \rho_m$ as $m \to +\infty$. For the second factor $\psi_2(w)$ it is easy to see that there exists a constant c_2 such that $|\psi_2(w)| = O(e^{c_2|w|})$ as $|w| \to +\infty$, because the integrands in $\psi_2(w)$ admit similar exponential estimates, uniform in $t \in \wp$, as $|w| \to +\infty$. Now (119) follows readily. $\square$

Lemma 9.3 *Let $\lambda = (p, q, r; a, b; x) \in \mathcal{F}^-$ be a solution to Problem* II.

(1) *Any pole of $f(w; \lambda)$ is simple and lies in the arithmetic progression $\{w_j := -j/r\}_{j=0}^{\infty}$. Conversely, $f(w; \lambda)$ actually has a pole at $w = w_j$ for every sufficiently large integer j, in particular it has infinitely many poles.*

(2) *$f(w; \lambda)$ has infinitely many zeros.*

(3) *$f(w; \lambda)$ and $\tilde{f}(w; \lambda)$ are linearly independent over the rational function field $\mathbb{C}(w)$.*

Proof. Assertion (1). It is evident from definition (3) that every pole of $f(w; \lambda)$ is simple and lies in the sequence $\{w_j\}_{j=0}^{\infty}$. By Lemma 4.1 of Iwasaki [8] we have

$$\operatorname*{Res}_{w=w_j} f(w) = C_j \cdot {}_2F_1(a_j,\, b_j;\, j + 2;\, x) \qquad (j = 0, 1, 2, \dots),$$

where $a_j := pw_j + j + a + 1$, $b_j := qw_j + j + b + 1$ and

$$C_j := \frac{(-1)^j}{r} \cdot \frac{(pw_j + a)_{j+1}\, (qw_j + b)_{j+1}}{j!\, (j + 1)!}\, x^{j+1}.$$

Since $\lambda \in \mathcal{F}^-$, we have $p < 0$, $q < 0$, $r > 0$ and $0 < x < 1$, so there exists an integer j_0 such that $pw_j + a > 0$, $qw_j + b > 0$ and hence $(-1)^j C_j > 0$ for every $j \geq j_0$. Notice also that if $j \geq j_0$ then $a_j > pw_j + a > 0$, $b_j > qw_j + b > 0$ and thus ${}_2F_1(a_j, b_j; j + 2; x) > 0$. Therefore $f(w)$ actually has a simple pole with a non-vanishing residue at $w = w_j$ for every $j \geq j_0$.

Assertion (2). Suppose the contrary that $f(w; \lambda)$ has at most finitely many zeros. Then it follows from assertion (1) that $u(w) := f(w; \lambda)/\Gamma(rw)$ is an entire holomorphic function with at most finitely many zeros. Here we have a uniform estimate $|1/\Gamma(rw)| = O\left(e^{r|w|(\log|w|+c_3)}\right)$ as $|w| \to \infty$, which follows from Stirling's formula: for any $\varepsilon \in (0, \pi)$ one has uniformly

$$\frac{1}{\Gamma(w)} \sim \frac{1}{\sqrt{2\pi}}\, w^{1/2}\, e^{w(1-\log w)} \qquad \text{as} \quad w \to \infty \quad \text{in} \quad |\arg w| < \pi - \varepsilon, \tag{120}$$

combined with the reflection formula (60) for the gamma function. By this estimate and Lemma 9.2 we have $|u(w)| \leq M_2\, e^{r\rho_m(\log \rho_m + c_4)}$ on the circle $|w| = \rho_m$ for every $m \geq m_0$. Given any $z \in \mathbb{C}$ with sufficiently large $|z|$, take an $m \geq m_0$ so that $\rho_{m-1} < |z| \leq \rho_m$. Since $u(w)$ is entire and $\rho_m < \rho_{m-1} + 2\sigma < |z| + 2\sigma$ by estimate (119), the maximum principle yields

$$|u(w)| \leq M_2\, e^{r\rho_m(\log \rho_m + c_4)} \leq M_2\, e^{r(|z|+2\sigma)\{\log(|z|+2\sigma)+c_4\}} \leq M_3\, e^{r|z|(\log|z|+c_5)},$$

which means that $u(w)$ is an entire function of order at most 1. Since $u(w)$ has at most finitely many zeros, Hadamard's factorization theorem allows us to write $u(w) = u_0(w)\, e^{c_6 w + c_7 w^2}$, where $u_0(w)$ is a nonzero polynomial (or constant) and c_6, $c_7 \in \mathbb{C}$, so that

$$\frac{u(w+1)}{u(w)} = \frac{u_0(w+1)}{u_0(w)} \cdot e^{c_6+c_7+2c_7 w} \sim e^{c_6+c_7+2c_7 w} \qquad \text{as} \quad w \to \infty.$$

On the other hand, since λ is a solution to Problem II, that is, satisfies condition (6), we have

$$\frac{u(w+1)}{u(w)} = \frac{f(w+1;\lambda)}{f(w;\lambda)} \cdot \frac{\varGamma(rw)}{\varGamma(r(w+1))} = R(w;\lambda) \cdot \frac{\varGamma(rw)}{\varGamma(r(w+1))}.$$

As a rational function, $R(w;\lambda) \sim M_4\, w^{k_1}$ as $w \to \infty$ for some $M_4 \in \mathbb{C}^{\times}$ and $k_1 \in \mathbb{Z}$, while by Stirling's formula (120) we have $\varGamma(rw)/\varGamma(r(w+1)) \sim (rw)^{-r}$ as $\mathbb{R} \ni w \to +\infty$. Thus

$$\frac{u(w+1)}{u(w)} \sim M_4\, r^{-r} \cdot w^{k_1 - r} \quad \text{and so} \quad w^{r-k_1} e^{2c_7 w} \sim M_4\, r^{-r} e^{-c_6 - c_7} \quad \text{as} \quad \mathbb{R} \ni w \to +\infty,$$

which forces $c_7 = 0$ and hence $u(w) = u_0(w)\, e^{c_6 w}$. As a polynomial, $u_0(w) \sim M_5\, w^{k_2}$ as $w \to \infty$ for some $M_5 \in \mathbb{C}^{\times}$ and $k_2 \in \mathbb{Z}_{\ge 0}$. It follows from estimates (119) and (120) that

$$\begin{aligned} 1 &= \left| \frac{u(\rho_m)}{u_0(\rho_m)\, e^{c_6 \rho_m}} \right| = \left| \frac{f(\rho_m;\lambda)}{u_0(\rho_m)\, e^{c_6\rho_m}} \cdot \frac{1}{\varGamma(r\rho_m)} \right| \lesssim \frac{M_1\, e^{c_1 \rho_m} \cdot (r\rho_m)^{1/2} e^{r\rho_m \{1 - \log(r\rho_m)\}}}{|M_5|\, \rho_m^{k_2} \cdot e^{c_6 \rho_m} \cdot \sqrt{2\pi}} \\ &= M_6 \cdot \rho_m^{\frac{1}{2} - k_2} \cdot e^{\rho_m (c_8 - r \log \rho_m)} \to 0 \qquad \text{as} \quad m \to \infty, \end{aligned}$$

where $M_6 := \frac{M_1}{|M_5|}\sqrt{\frac{r}{2\pi}}$ and $c_8 := c_1 - c_6 + r - r\log r$. This contradiction shows that $f(w;\lambda)$ must have infinitely many zeros.

Assertion (3) can be established by modifying the proof of Lemma 11.1 in Iwasaki [8]. Suppose the contrary that $f(w;\lambda)$ and $\tilde{f}(w;\lambda)$ are linearly dependent over $\mathbb{C}(w)$, so that there exists a rational function $T(w) \in \mathbb{C}(w)$ such that $\tilde{f}(w;\lambda) = T(w) f(w;\lambda)$, because $f(w;\lambda)$ does not vanish identically. Then as in the proof of [8, Lemma 11.1] we have

$$f(w;\lambda) f_1(w;\lambda) = (1-x)^{(r-p-q)w - a - b - 1}, \tag{121}$$

where $f_1(w;\lambda)$ is defined by

$$\begin{aligned} f_1(w;\lambda) := {}& {}_2F_1((p-r)w + a + 1, (q-r)w + b + 1; 1 - rw; x) \\ & + \frac{(pw+a)(qw+b)x}{rw(rw-1)} \cdot T(w) \cdot {}_2F_1((p-r)w + a + 1, (q-r)w + b + 1; 2 - rw; x). \end{aligned}$$

Observe that any pole of $f_1(w;\lambda)$ is either a pole of $T(w)$ or in the discrete set $\frac{1}{r}\mathbb{Z}_{\ge 0}$. Thus $f_1(w;\lambda)$ cannot have infinitely many poles off the positive real axis $\mathbb{R}_+$. On the other hand, applying Euler's transformation (15b) to definition (3), we have

$$f(w;\lambda) = (1-x)^{(r-p-q)w-a-b}{}_2F_1((r-p)w - a, (r-q)w - b; rw; x),$$

where $r - p > 0$, $r - q > 0$ and $r > 0$ while $0 < x < 1$ by the assumption $\lambda \in \mathcal{F}^-$. Thus

$$f(w;\lambda) > 0 \qquad \text{for every real} \quad w > \max\{a/(r-p),\, b/(r-q),\, 0\}, \tag{122}$$

so $f(w;\lambda)$ cannot have infinitely many zeros on $\mathbb{R}_+$. Assertion (2) then implies that $f(w;\lambda)$ must have infinitely many zeros off $\mathbb{R}_+$. Therefore $f(w;\lambda)$ admits a zero $w_0 \in \mathbb{C} \setminus \mathbb{R}_+$ that is not a pole of $f_1(w;\lambda)$. Substituting $w = w_0$ into equation (121) yields a contradiction $0 = f(w_0;\lambda) f_1(w_0;\lambda) = (1-x)^{(r-p-q)w_0-a-b} \neq 0$, which shows that $f(w;\lambda)$ and $\tilde{f}(w;\lambda)$ are linearly independent over the rational function field $\mathbb{C}(w)$. □

Proposition 9.4 *Let $\lambda = (p,q,r;a,b;x) \in \mathcal{F}^-$ in what follows.*

(1) *Any solution to Problem* I *or* II *in $\mathcal{F}^-$ is non-elementary.*

(2) *Any integral solution in $\mathcal{F}^-$ to Problem* II *comes from contiguous relations.*

(3) *For any integral solution $\lambda \in \mathcal{F}^-$ to Problem* II*, its reciprocal $\check{\lambda}$ is an* (A)*-solution in $\mathcal{D}^-$ so that $r \equiv 0 \mod 2$ and $\lambda = \lambda^{\vee\vee}$ is the reciprocal of an* (A)*-solution $\check{\lambda} \in \mathcal{D}^-$.*

(4) *Any rational solution in $\mathcal{F}^-$ to Problem* II *essentially comes from contiguous relations.*

Proof. Assertion (1) follows directly from assertion (1) of Lemma 9.3. With assertion (3) of Lemma 9.3 the proof of assertion (2) is exactly the same as that of [8, Proposition 11.2]. The proof of assertion (3) proceeds as follows. Since the λ in assertion (3) comes from contiguous relations by assertion (2), Lemma 6.1 can be used to infer that its reciprocal $\check{\lambda}$ is a solution to Problem II and hence becomes an (A)-solution in $\mathcal{D}^-$ by Theorems 3.1 and 3.2. In other words, $\lambda = \lambda^{\vee\vee}$ is the reciprocal of an (A)-solution $\check{\lambda} \in \mathcal{D}^-$. Applying assertion (2) of Theorem 3.2 to $\check{\lambda}$ we have $r = \check{r} - \check{p} - \check{q} \equiv 0 \mod 2$. Assertion (4) is an immediate consequence of assertion (2) applied to a multiplication $k\lambda$, where $k \in \mathbb{N}$ is chosen so that $k\lambda$ becomes integral. □

9.2 Gamma Product Formulas

Now that Proposition 9.4 is established, the reciprocity can be used to tackle Problem 1.1 for *rational* data in $\mathcal{F}^-$ through their suitable multiplications.

Lemma 9.5 *If $\lambda = (p,q,r;a,b;x) \in \mathcal{F}^-$ is a rational solution to Problem* II *with*

$$\frac{f(w+1;\lambda)}{f(w;\lambda)} = d \frac{(w+u_1)\cdots(w+u_m)}{(w+v_1)\cdots(w+v_n)}, \tag{123}$$

for some $d \in \mathbb{C}^\times$, $u_1, \dots, u_m \in \mathbb{C}$ and $v_1, \dots, v_m \in \mathbb{C}$, then $m = n$, d is a positive number given by formula (40)*, and λ becomes a solution to Problem* I *with gamma product formula*

$$f(w;\lambda) = C \cdot d^w \cdot \frac{\Gamma(w+u_1)\cdots\Gamma(w+u_m)}{\Gamma(w+v_1)\cdots\Gamma(w+v_m)}, \tag{124}$$

where C is a positive constant.

Proof. Let $k \in \mathbb{N}$ be the least common denominator of p, q, $r \in \mathbb{Q}$. Then the multiplication $k\lambda = (kp,kq,kr;a,b;x) \in \mathcal{F}^-$ is an integral solution to Problem II. It follows from assertion (3) of Proposition 9.4 that $(k\lambda)^\vee$ is an (A)-solution in $\mathcal{D}^-$ and $k\lambda = (k\lambda)^{\vee\vee}$ is its reciprocal. Accordingly we can apply Proposition 7.6 to the (A)-solution $(k\lambda)^\vee = (k|p|, k|q|, k(r-p-q); \check{a}, \check{b}; \check{x}) \in \mathcal{D}^-$. Adapting GPF (108) to the current situation, we have

$$f(w;k\lambda) = f\left(w;(k\lambda)^{\vee\vee}\right) = \check{C} \cdot \delta^{kw} \cdot \frac{\prod_{i=0}^{kr-1} \Gamma\left(w + \frac{i}{kr}\right)}{\prod_{i=1}^{kr} \Gamma(w+\xi_i)}, \tag{125}$$

where $r-p-q$ in the general formula (108) is now kr, the constant $\check{d}$ there is now δ^k with

$$\delta := r^r \sqrt{\frac{|p|^{|p|}\, |q|^{|q|}\, (1-x)^{r-p-q}}{(r-p)^{r-p}\, (r-q)^{r-q}\, x^r}}, \tag{126}$$

which is different from δ in (59), while the numbers $\check{v}_i$ there are now written ξ_i. Under these circumstances the conditions (111) and (112) with λ replaced by $(k\lambda)^\vee$ are represented as

$$\xi_1 + \cdots + \xi_{kr} = \frac{kr-1}{2}, \tag{127}$$

$$\xi_1, \dots, \xi_{kr} \in \mathbb{Q}; \qquad \frac{c}{k} \le \xi_1, \dots, \xi_{kr} < \frac{c}{k} + 1, \qquad c = c(\lambda) := \frac{1-a-b}{r-p-q}, \tag{128}$$

where we used $c\left((k\lambda)^\vee\right) = -c(k\lambda) = -c(\lambda)/k = -c/k$ to derive condition (128).

On the one hand formula (125) leads to a closed-form expression

$$\frac{f(w+1;k\lambda)}{f(w;k\lambda)} = \delta^k \cdot \frac{\prod_{i=0}^{kr-1}\left(w + \frac{i}{kr}\right)}{\prod_{i=1}^{kr}(w+\xi_i)}, \tag{129}$$

whereas on the other hand applying formula (14) to assumption (123) yields

$$\frac{f(w+1;k\lambda)}{f(w;k\lambda)} = d^k \cdot k^{k(m-n)} \cdot \prod_{i=0}^{k-1} \frac{\left(w+\frac{u_1+i}{k}\right)\cdots\left(w+\frac{u_m+i}{k}\right)}{\left(w+\frac{v_1+i}{k}\right)\cdots\left(w+\frac{v_n+i}{k}\right)}. \tag{130}$$

Since (129) and (130) must be exactly the same, they must be so asymptotically, that is, $\delta^k \sim d^k \cdot k^{k(m-n)} \cdot w^{k(m-n)}$ as $w \to \infty$, which forces $m = n$ and $\delta^k = d^k$. Formula (123) together with (122) and $m = n$ yields $0 < f(w+1;\lambda)/f(w;\lambda) \to d$ as $\mathbb{R} \ni w \to +\infty$ and so $d \ge 0$. Since δ is positive by definition (126), equation $\delta^k = d^k$ gives $d = \delta$. Thus in view of definition (126), d must be given by formula (40). The coincidence of (129) and (130) yields

$$\frac{\prod_{i=0}^{kr-1}\left(w + \frac{i}{kr}\right)}{\prod_{i=1}^{kr}(w+\xi_i)} = \prod_{i=0}^{k-1} \frac{\left(w+\frac{u_1+i}{k}\right)\cdots\left(w+\frac{u_m+i}{k}\right)}{\left(w+\frac{v_1+i}{k}\right)\cdots\left(w+\frac{v_m+i}{k}\right)},$$

so that gamma product formula (125) can be recast to

$$f(w;k\lambda) = \check{C} \cdot d^{kw} \cdot \prod_{i=0}^{k-1} \frac{\Gamma\left(w+\frac{u_1+i}{k}\right)\cdots\Gamma\left(w+\frac{u_m+i}{k}\right)}{\Gamma\left(w+\frac{v_1+i}{k}\right)\cdots\Gamma\left(w+\frac{v_m+i}{k}\right)}, \tag{131}$$

where $m = n$ and $d = \delta$ are also incorporated. Replacing w by w/k in (131) and using the multiplication formula (12) for the gamma function, we obtain formula (124) with $C = \check{C} \cdot k^{v-u}$, where $u := u_1 + \cdots + u_m$ and $v := v_1 + \cdots + v_m$ are real numbers as λ is a real data. Since $\check{C}$ is positive, so is the constant C. $\square$

Proposition 9.6 *If $\lambda = (p, q, r; a, b; x) \in \mathcal{F}^-$ is a rational solution to Problem* $\mathbb{I}$, *then $r \in \mathbb{N}$, $a, b \in \mathbb{Q}$, x algebraic, and λ is a solution to Problem* I *with GPF* (39) *where C is a positive constant, d is given by formula* (40) *and $v_1, \dots, v_r$ satisfy condition* (41).

Proof. By assertion (1) of Lemma 9.3 the function $f(w;\lambda)$ has infinitely many poles so we must have $m \ge 1$ in formula (124). There exists an integer s with $0 \le s \le m$ such that

$$\text{(i)} \quad u_i - v_j \notin \mathbb{Z} \quad \text{for any} \quad i, j = 1, \dots, s, \qquad \text{(ii)} \quad u_i - v_i \in \mathbb{Z} \quad \text{for any} \quad i = s+1, \dots, m,$$

after suitable rearrangements of $\{u_1, \dots, u_m\}$ and $\{v_1, \dots, v_m\}$, where condition (i) resp. (ii) should be ignored if $s = 0$ resp. $s = m$. In view of property (ii) a repeated use of the recursion formula $\Gamma(w+1) = w\Gamma(w)$ allows us to rewrite formula (124) as

$$f(w;\lambda) = S(w) \cdot d^w \cdot \frac{\Gamma(w+u_1)\cdots\Gamma(w+u_s)}{\Gamma(w+v_1)\cdots\Gamma(w+v_s)}, \tag{132}$$

where $S(w)$ is a nontrivial rational function of w. We must have $s \ge 1$, for otherwise $f(w;\lambda) = S(w) \cdot d^w$ could not have infinitely many poles, contradicting assertion (1) of Lemma 9.3.

Take a nonnegative integer i sufficiently large so that neither $\omega_0 := -u_1 - i$ nor $\omega_1 := -u_1 - i - 1$ are zeros of $S(w)$. Then by property (i), $w = \omega_0$ and $w = \omega_1$ are actually poles of $f(w;\lambda)$, so by assertion (1) of Lemma 9.3 there exist nonnegative integers j_0 and j_1 with $j_0 < j_1$ such that $\omega_0 = -j_0/r$ and $\omega_1 = -j_1/r$. Thus we have $1 = \omega_0 - \omega_1 = (j_1 - j_0)/r$, namely, $r = j_1 - j_0 \in \mathbb{N}$ and hence the number k in the proof of Lemma 9.5 is the least common denominator of p, $q \in \mathbb{Q}$. The assertion that a, $b \in \mathbb{Q}$ and x is rational follows from the first part of Remark 7.7 applied to $(k\lambda)^\vee$ in place of λ.

Again by assertion (1) of Lemma 9.3, for a sufficiently large j_2, the set

$$\left\{-\frac{j}{r}\right\}_{j \ge j_2 r} = \coprod_{i=1}^{r} \left\{-j - \frac{i-1}{r}\right\}_{j \ge j_2}$$

constitutes all but a finite number of poles of $f(w;\lambda)$. The same is true with the multi-set

$$\bigcup_{i=1}^{s} \{-j - u_i\}_{j \ge j_3} \qquad \text{(union as multi-sets)},$$

due to formula (132) and property (i). Since all poles are simple, we have $u_i - u_j \notin \mathbb{Z}$ for every distinct $i, j = 1, \dots, s$, so the union of multi-sets above is just a disjoint union of ordinary sets. As the two sets above can differ only by a finite number of elements, we must have

$$r = s, \qquad u_i - (i-1)/r \in \mathbb{Z} \qquad (i = 1, \dots, r),$$

after taking a suitable rearrangement of $u_1, \dots, u_r$. Thus property (i) is equivalent to

$$v_1, \dots, v_r \notin \frac{1}{r}\mathbb{Z}, \tag{133}$$

and a further repeated use of the recursion formula $\Gamma(w+1) = w\Gamma(w)$ converts (132) to

$$f(w;\lambda) = S(w) \cdot d^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(w + \frac{i}{r}\right)}{\prod_{i=1}^{r} \Gamma(w + v_i)}, \tag{134}$$

with a possibly different rational function $S(w)$ of w, where we may assume

$$c \le \operatorname{Re} v_1, \dots, \operatorname{Re} v_r < c + 1, \qquad c := \frac{1 - a - b}{r - p - q}, \tag{135}$$

after translating $v_1, \dots, v_r$ by suitable integers and making yet another use of the recursion formula $\Gamma(w+1) = w\Gamma(w)$ with an ensuing modification of the rational function $S(w)$.

Replacing w by kw in formula (134) and using formulas (11) and (12), we have

$$f(w; k\lambda) = k^{(r-1)/2-v} \cdot S(kw) \cdot d^{kw} \cdot \frac{\prod_{i=0}^{kr-1} \Gamma\left(w + \frac{i}{kr}\right)}{\prod_{i=1}^{kr} \Gamma(w + \eta_i)}, \tag{136}$$

where $v := v_1 + \cdots + v_r$ and $\eta_1, \dots, \eta_{kr}$ are the numbers defined by the multi-set

$$\{\eta_1, \dots, \eta_{kr}\} := \{v_{i,j} \,:\, i = 1, \dots, r,\ j = 0, \dots, k-1\}, \qquad v_{i,j} := \frac{v_i + j}{k}. \tag{137}$$

In view of this definition the estimate (135) leads to

$$\frac{c}{k} \le \operatorname{Re} \eta_1, \dots, \operatorname{Re} \eta_{kr} < \frac{c}{k} + 1. \tag{138}$$

Comparing formulas (125) and (136) with $\delta = d$ taken into account, we find that

$$\frac{\prod_{i=1}^{kr} \Gamma(w + \eta_i)}{\prod_{i=1}^{kr} \Gamma(w + \xi_i)} = \left(\check{C}\right)^{-1} k^{(r-1)/2-v} \cdot S(kw),$$

which must be a rational function of w, having only at most finitely many poles and zeros. This forces $\eta_i - \xi_i \in \mathbb{Z}$ for every $i = 1, \dots, kr$, after a suitable rearrangement of $\xi_1, \dots, \xi_{kr}$. But in view of conditions (128) and (138), this coincidence modulo $\mathbb{Z}$ must be an exact coincidence

$$\eta_i = \xi_i \qquad (i = 1, \dots, kr). \tag{139}$$

So $\left(\check{C}\right)^{-1} k^{(r-1)/2-v} \cdot S(kw) = 1$ and thus $S(w) = \check{C} k^{v-(r-1)/2} =: C$ must be a positive constant and formula (134) reduces to GPF (39). We can arrange $\eta_1, \dots, \eta_{kr}$ so that $\eta_i = v_{i,0} = v_i/k$ for each $i = 1, \dots, r$. Then coincidence (139) and the rationality in (128) imply $v_i = k\eta_i = k\xi_i \in \mathbb{Q}$ for every $i = 1, \dots, r$. This together with (133) and (135) gives $v_1, \dots, v_r \in \left(\mathbb{Q} \setminus \frac{1}{r}\mathbb{Z}\right) \cap [c, c+1)$ in conditions (41). To prove the remaining condition in (41) we observe that

$$\sum_{i=1}^{kr} \xi_i = \sum_{i=1}^{kr} \eta_i = \sum_{i=1}^{r} \sum_{j=0}^{k-1} v_{i,j} = \sum_{i=1}^{r} \sum_{j=0}^{k-1} \frac{v_i + j}{k} = \sum_{i=1}^{r} v_i + \frac{(k-1)r}{2}, \tag{140}$$

where we used (139) in the first equality and (137) in the second and third equalities. Comparing (140) with (127) we have $v_1 + \cdots + v_r = (r-1)/2$ and hence proves conditions (41). $\Box$

It is evident that Theorem 3.12 follows from Propositions 9.4 and 9.6. The proof of Theorem 3.14 is virtually contained in the proof of Proposition 9.6 as a special case $k = 2$; the only necessary modification is to deduce the condition $0 \le v_1, \dots, v_r < 1$ for a (B)-solution from a similar condition for an (A)-solution.

10 From an Algorithmic Point of View

We put all the results on $\mathcal{D}^- \cup \mathcal{F}^-$ (in §3) into context from an algorithmic point of view and discuss how to enumerate all (rational) solutions $\lambda = (p, q, r; a, b, x)$ with a prescribed value of $\boldsymbol{p} = (p, q; r)$, or perhaps how to prove the nonexistence of such solutions, both in finite steps.

10.1 Integral Solutions

If $\lambda = (p, q, r; a, b; x)$ is an (A)-solution in $\mathcal{D}^-$ then $\boldsymbol{p} = (p, q; r) \in D_{\mathrm{A}}^-$ by assertion (2) of Theorem 3.2, where D_{A}^- is defined by formula (22). Putting Theorems 3.1, 3.3, 3.6 and 3.8 together along with Theorem 10.1 below, we are able to develop an algorithm to find all (A)-solutions $\lambda \in \mathcal{D}^-$ with a prescribed value of $\boldsymbol{p} \in D_{\mathrm{A}}^-$. To describe it we introduce some notation. Let $\langle \varphi(z) \rangle_k := \sum_{j=0}^{k} c_j\, z^j$ be the *truncation* at degree k of a power series $\varphi(z) = \sum_{j=0}^{\infty} c_j\, z^j$. For a data $\lambda \in \mathcal{D}^-$ with $\boldsymbol{p} \in D_{\mathrm{A}}^-$ we consider two "truncated hypergeometric products":

$$V(w; \lambda) := (rw)_{r-1} \, \langle \, {}_2F_1(\boldsymbol{\alpha}^*(w); z) \cdot {}_2F_1(\boldsymbol{v} - \boldsymbol{\alpha}^*(w+1); z) \, \rangle_k \big|_{z=x},$$

$$P(w; \lambda) := (rw)_{r} \, \langle \, {}_2F_1(\boldsymbol{\alpha}^*(w); z) \cdot {}_2F_1(\boldsymbol{1} - \boldsymbol{\alpha}^*(w+1); z) \, \rangle_k \big|_{z=x},$$

where $(s)_n := \Gamma(s+n)/\Gamma(s)$, $\boldsymbol{\alpha}^*(w) := ((r-p)w - a,\, (r-q)w - b;\, rw)$, $\boldsymbol{v} := (1, 1; 2)$, $\boldsymbol{1} := (1, 1; 1)$ and $k := \max\{r - p - 1,\, r - q - 1\}$. Note that $V(w; \lambda)$ is written $\Phi(w; \lambda)$ in the previous paper [8] but the latter symbol is already used in a different context in §4 of the present article and hence altered. It is shown in [8, Lemma 11.10] that $V(w; \lambda)$ is a polynomial of degree at most $r - 1$ in w and so admits an expansion

$$V(w; \lambda) = \sum_{\nu=0}^{r-1} V_\nu(\lambda)\, w^\nu = \sum_{\nu=0}^{r-1} V_\nu(a, b; x)\, w^\nu,$$

where $V_\nu(\lambda)$ is written $V_\nu(a, b; x)$ when $\boldsymbol{p}$ is understood to be a priori given. It is also known that $P(w; \lambda)$ is a polynomial of degree at most r. Our algorithm is based on the following.

Theorem 10.1 *$\lambda \in \mathcal{D}^-$ is a solution of type* (A) *if and only if $\boldsymbol{p} \in D_{\mathrm{A}}^-$ and $V(w; \lambda)$ vanishes identically as a polynomial of w, that is, $(a, b; x)$ is a simultaneous root of algebraic equations*

$$V_\nu(a, b; x) = 0 \qquad (\nu = 0, \dots, r-1), \tag{141}$$

in which case $P(w; \lambda)$ is exactly of degree r in w and $R(w; \lambda)$ in formula (6) *is given by*

$$R(w; \lambda) = (1 - x)^{r-p-q-1} \cdot \frac{(rw)_r}{P(w; \lambda)}. \tag{142}$$

This theorem follows from [8, Theorems 2.9 and 2.10]. Now we have the following.

Algorithm 10.2 To enumerate all (A)-solutions $\lambda \in \mathcal{D}^-$ with any prescribed $\boldsymbol{p} \in D_{\mathrm{A}}^-$:

(1) Check if $\boldsymbol{p} = (p, q; r)$ satisfies division relation (28) in Theorem 3.6.

(2) If it is alright, then following Theorem 3.8 find a candidate for $\boldsymbol{a} = (a, b)$ explicitly in terms of $\boldsymbol{p}$, which must be in one of Cases 1–6 of Table 1.

(3) Substitute the ensuing candidate $\boldsymbol{a}$ into (141) to derive algebraic equations for x over $\mathbb{Q}$:

$$V_\nu(x) = 0 \qquad (\nu = 0, \dots, r-1), \tag{143}$$

where $V_{r-1}(x) = 0$ is equivalent to equation (23) in Theorem 3.3 (see [8, Remark 2.11]).

(4) Check if equations (143) admits a simultaneous root x with $0 < x < 1$. If so, we certainly get an (A)-solution $\lambda = (\boldsymbol{p}; \boldsymbol{a}; x) \in \mathcal{D}^-$; otherwise the current candidate gives no solution.

(5) If we actually have a solution λ in step (4), put it into formula (142) to find $R(w; \lambda)$ explicitly. If $R(w; \lambda)$ is of the form (7) (inevitably with $m = n$), then we have a GPF (5) by Theorem 3.1, where the constant C can be determined explicitly by evaluating $f(w; \lambda)$ at such a point $w = w_0$ that the hypergeometric series $f(w_0; \lambda)$ is terminating or more specifically it is vanishing except for its leading term.

(6) Repeat the procedures (2)–(5) until all candidates are exhausted. By Theorem 3.10 there are only a finite number of candidates so that the algorithm terminates in finite steps.

Empirically, it almost surely occurs that algebraic equations (143) have no roots in common for all candidates, in which case Algorithm 10.2 serves as a rigorous proof of the nonexistence of any solution in $\mathcal{D}^-$ with a given value of $\boldsymbol{p} \in D_{\mathrm{A}}^-$. Observe also that for any $\boldsymbol{p} = (p, q; r) \in D_{\mathrm{A}}^-$ with division relation (28) there is an estimate $1 \le p,\, q \le r/2$. Thus by the use of Algorithm 10.2 the enumeration of all (A)-solutions in $\mathcal{D}^-$ with any prescribed bound for r terminates in finite steps after producing at most finitely many solutions. In view of estimate (116) in Lemma 8.4, this is true even with any prescribed bound for $\check{r} := r - p - q$. For example, there are exactly seven (A)-solutions in $\mathcal{D}^-$ with $\check{r} = 2$, all of which are contained in [8, Table 2].

We proceed to the treatment of integral solutions in $\mathcal{F}^-$, which is based on the following.

Theorem 10.3 *Reciprocity* (20) *induces a bijection between the set of all* (A)*-solutions in* $\mathcal{D}^-$ *and the set of all integral solutions in* $\mathcal{F}^-$.

This theorem is an immediate consequence of Theorems 3.11 and 3.12. It implies that all integral solutions $\lambda = (p, q, r; a, b; x) \in \mathcal{F}^-$ with a given $\boldsymbol{p} = (p, q; r)$ is in one-to-one correspondence with all (A)-solutions $\check{\lambda} = (\check{p}, \check{q}, \check{r}; \check{a}, \check{b}; \check{x}) \in \mathcal{D}^-$ with a given $\check{\boldsymbol{p}} = (\check{p}, \check{q}; \check{r}) = (-p, -q; r - p - q) = (|p|, |q|; r + |p| + |q|)$, so that the enumeration of former solutions is accomplished through the enumeration of latter solutions by the use of Algorithm 10.2.

Corollary 10.4 *There are at most finitely many integral solutions* $\lambda = (p, q, r; a, b; x) \in \mathcal{F}^-$ *with any prescribed bound for* r*. Moreover any integral solution* $\lambda \in \mathcal{F}^-$ *must satisfy*

$$-3r \le p,\, q \le -1, \qquad -5r \le p + q \le -2.$$

These inequalities follow from estimates (116) in Lemma 8.4 by replacing λ with $\check{\lambda}$.

10.2 Rational Solutions

We turn our attention to dealing with rational solutions in $\mathcal{D}^- \cup \mathcal{F}^-$ that are non-integral. In $\mathcal{D}^-$ this amounts to considering solutions of type (B), since all solutions in $\mathcal{D}^-$ are rational and the dichotomy of types (A) and (B) is exactly that of 'integral' and 'non-integral'.

A solution $\lambda = (p, q, r; a, b; x)$ to Problem I or II is said to be *divisible* by an integer $k \ge 2$, if the data $\lambda/k := (p/k, q/k, r/k; a, b; x)$ is also a solution to the same problem, in which case λ/k is referred to as the *division* of λ by k, in particular, $\lambda/2$ is a *half* of λ.

Lemma 10.5 *For an integer $k \ge 2$, an integral solution $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^- \cup \mathcal{F}^-$ is divisible by k if and only if $k|r$ and there exist $d > 0$ and $v_1, \dots, v_s \in \mathbb{Q}$ with $s := r/k$ such that*

$$R(w;\lambda) = d \cdot \frac{\prod_{i=0}^{r-1} \left(w + \frac{i}{r}\right)}{\prod_{i=1}^{s} \prod_{j=0}^{k-1} \left(w + \frac{v_i + j}{k}\right)}. \tag{144}$$

Proof. Essential is the "if" part. Since Problems I and II are equivalent for integral (or more generally rational) data in $\mathcal{D}^- \cup \mathcal{F}^-$ by Theorem 3.1 and assertion (3) of Theorem 3.12, the closed-form condition (144) lifts to a "multiplied" gamma product formula

$$f(w;\lambda) = C \cdot d^w \cdot \frac{\prod_{i=0}^{r-1} \Gamma\left(w + \frac{i}{r}\right)}{\prod_{i=1}^{s} \prod_{j=0}^{k-1} \Gamma\left(w + \frac{v_i+j}{k}\right)} = C \cdot d^w \cdot \prod_{j=0}^{k-1} \frac{\prod_{i=0}^{s-1} \Gamma\left(w + \frac{(i/s)+j}{k}\right)}{\prod_{i=1}^{s} \Gamma\left(w + \frac{v_i+j}{k}\right)},$$

which in turn descends through the multiplication formula (12) to a GPF

$$f(w;\lambda/k) = f(w/k;\lambda) = C \cdot d^{w/k} \cdot \frac{\prod_{i=0}^{s-1} \Gamma\left(w + \frac{i}{s}\right)}{\prod_{i=1}^{s} \Gamma\left(w + v_i\right)},$$

with a possibly different C. Thus λ/k is a solution and hence λ is divisible by k. The proof of "only if" part is a simple reversing of the argument so far, with the positivity $d > 0$ coming from formulas (25) and (40), while the rationality $v_1, \dots, v_s \in \mathbb{Q}$ from properties (32) and (41). □

By assertions (2) and (4) of Theorems 3.2 and assertion (3) of Theorem 3.12,

(1) any (B)-solution in $\mathcal{D}^-$ is a *half* of an (A)-solution $\lambda_* = (p_*, q_*, r_*; a_*, b_*; x_*) \in \mathcal{D}^-$ such that p_* and q_* are odd positive integers and r_* is an even positive integer,

(2) any rational solution in $\mathcal{F}^-$ is the division λ_*/k of an integral solution $\lambda_* \in \mathcal{F}^-$ by a *divisor* k of r_*, where r_* is necessarily an even positive integer.

Thus finding a rational solution consists of finding an integral solution λ_* by using the recipe in §10.1 and verifying if λ_* is divisible by a divisor of r_* based on Lemma 10.5. All rational solutions are obtained in this manner. Note that any seed solution λ_* produces only a finite number of division solutions. All (B)-solutions $\lambda = (p,q,r;a,b;x) \in \mathcal{D}^-$ with any prescribed bound for r are only of a finite cardinality, since only division by 2 is involved in this case. It is not known whether this is also the case for rational solutions in $\mathcal{F}^-$, because r_* and k may be arbitrary large while $r = r_*/k$ is kept bounded (see Problem 11.4).

10.3 Examples of Solutions in the South-West Domain

Some examples of (A)-solutions in $\mathcal{D}^-$ were presented in the previous article [8, Table 2]. Their reciprocal solutions in $\mathcal{F}^-$ are given in the corresponding places of Table 4, which exhibits for each solution the data $\lambda = (p,q,r;a,b;x) \in \mathcal{F}^-$ itself as well as the values of the numbers d and $v_1, \dots, v_r$ in GPF (39), with a brief remark in the last column, if any. Solutions in the top and second row of Table 4 are Gosper's conjectural identities in Gessel and Stanton [7, formulas (6.6) and (6.5)], proved later by Koepf [9, Table 5] using an extension of Wilf-Zeilberger method [14]. Self-dual solutions are indicated so in the remark column, while any

r	p	q	x	d	a	b	v_1	v_2	v_3	v_4	remark
2	-1	-1	$\frac{1}{9}$	$\frac{2^8}{3^5}$	$\frac{11}{8}$	$\frac{9}{8}$	$\frac{5}{24}$	$\frac{7}{24}$			GS [7, (6.6)]
					$\frac{5}{8}$	$\frac{7}{8}$	$\frac{1}{24}$	$\frac{11}{24}$			GS [7, (6.5)]
					$\frac{5}{4}$	$\frac{3}{4}$	$\frac{1}{12}$	$\frac{5}{12}$			self-dual
4	-1	-1	$\frac{1}{5}$	$\frac{2^{14}}{5^6}$	$\frac{9}{8}$	$\frac{5}{8}$	$\frac{3}{40}$	$\frac{7}{40}$	$\frac{23}{40}$	$\frac{27}{40}$	divisible by 2
					$\frac{3}{8}$	$\frac{7}{8}$	$\frac{1}{40}$	$\frac{9}{40}$	$\frac{21}{40}$	$\frac{29}{40}$	divisible by 2
2	-2	-2	$\frac{1}{4}(3\sqrt{3}-5)$	$\frac{3^4}{2^7}\sqrt{3}$	$\frac{5}{3}$	$\frac{4}{3}$	$\frac{1}{12}$	$\frac{5}{12}$			self-dual
	-3	-1	$9-4\sqrt{5}$	$\frac{2^8}{5^3}(5-2\sqrt{5})$	$\frac{9}{4}$	$\frac{5}{4}$	$\frac{3}{20}$	$\frac{7}{20}$			
					$\frac{7}{4}$	$\frac{3}{4}$	$\frac{1}{20}$	$\frac{9}{20}$			
	-4	-2	$17-12\sqrt{2}$	$\frac{2^{10}}{3^3}(17-12\sqrt{2})$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{12}$	$\frac{5}{12}$			self-dual

Table 4: Some integral solutions in $\mathcal{F}^-$; reciprocals of the (A)-solutions in [8, Table 2].

pair of consecutive solutions without this indication is a dual pair of solutions. Two solutions with $(p, q; r) = (-1, -1; 4)$ are divisible by 2, halves of which are tabulated in Table 5. These two solutions in $\mathcal{F}^-$ are "dual" to each other as well as "reciprocal" to the (B)-solutions in [8, Table 3], although we have to be careful in applying these terminologies to non-integral solutions (as mentioned in Remark 4.2).

11 Concluding Discussions

The main orientation of the field has been toward searching for as many solutions as possible, but the converse orientation of confining solutions into as slim a region as possible or perhaps of proving the nonexistence of solutions other than those already known is equally important for the ultimate goal of a complete enumeration. In the latter direction there are such interesting studies as [11, 15, 16]. The principal aim of our studies is also in this direction. With this in mind we close this article by giving a couple of problems that should be discussed in the future.

A solution to Problem I or II is said to be *primitive* if it is a multiplication of no other

r	p	q	x	d	a	b	v_1	v_2
2	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{5}$	$\frac{2^7}{5^3}$	$\frac{9}{8}$	$\frac{5}{8}$	$\frac{3}{20}$	$\frac{7}{20}$
					$\frac{3}{8}$	$\frac{7}{8}$	$\frac{1}{20}$	$\frac{9}{20}$

Table 5: Two non-integral rational solutions in $\mathcal{F}^-$.

solution. Since any solution is a multiplication of a primitive solution, the enumeration of all solutions boils down to that of primitive ones. As a milestone toward the complete enumeration we pose the following problem which is still very ambitious.

Problem 11.1 Are there infinitely many *primitive* solutions in domain $\mathcal{D}^-$ or only a finite number of them ? Ask the same question for domain $\mathcal{E}^{*-}$ or region $\mathcal{I}^{*-}$ or anywhere else.

We have no idea as to whether the cardinality of them is finite or infinite. As r becomes larger, it is increasingly more difficult that the r algebraic equations (143) have a root in common, but there is no logical reasoning that prohibits this miracle. Either answer is welcome as it contributes to the topic in one or the other orientation. Solving Problem 11.1 seems to require an amazingly new idea.

For solutions in $\mathcal{D}^-$ such a variety of results are at our disposal as Theorems 3.2, 3.3, 3.6–3.8, 3.10, 3.11, 3.14, while only Proposition 3.5 and Theorem 3.15 are available for solutions in $\mathcal{E}^{*-}$ besides Theorem 3.1 which is common with $\mathcal{D}^-$ and $\mathcal{E}^{*-}$.

Problem 11.2 Lift our understanding of solutions in $\mathcal{E}^{*-}$ to the same level as that of solutions in $\mathcal{D}^-$ by establishing appropriate $\mathcal{E}^{*-}$-counterparts of the above-mentioned results on $\mathcal{D}^-$.

For any solution $\lambda = (p, q, r; a, b; x) \in \mathcal{D}^-$ the numbers a and b must be rational by Theorems 3.10 and 3.14. This remains true for any rational solution in $\mathcal{F}^-$ by Theorem 3.12. However, it is not always the case in the south domain $\mathcal{E}^{*-}$. Indeed, for each pair (i, j) of integers with $j > k > 0$, Iwasaki [8, formula (123)] gives a one-parameter family of solutions $\lambda = (p, q, r; a, b; x) = (j-k, k-j, j+k; c, 1-c; 1/2) \in \mathcal{E}^{*-}$ with gamma product formula

$$\begin{aligned} &{}_2F_1(\,(j-k)w + c,\ -(j-k)w + 1 - c;\ (j+k)w;\ 1/2) \\ &\quad = \frac{\sqrt{2}\, k^{c/2}}{j^{(c-1)/2}\,(j+k)^{1/2}} \left\{ \frac{(j+k)^{j+k}}{2^{j+k}\, j^j\, k^k} \right\}^w \frac{\prod_{\nu=0}^{j+k-1} \Gamma\big(w + \frac{\nu}{j+k}\big)}{\prod_{\nu=0}^{j-1} \Gamma\big(w + \frac{c}{2j} + \frac{\nu}{j}\big) \prod_{\nu=0}^{k-1} \Gamma\big(w + \frac{1-c}{2k} + \frac{\nu}{k}\big)}, \end{aligned} \tag{145}$$

where c is a free parameter. If c is irrational, so are $a = c$ and $b = 1 - c$.

Problem 11.3 In spite of the existence of such solutions as (145), it might be expected that a and b must be rational for "almost all" solutions in $\mathcal{E}^{*-}$. Is this true and in what sense ?

Thanks to reciprocity our knowledge of integral solutions in $\mathcal{F}^-$ is at the same level as that of (A)-solutions in $\mathcal{D}^-$. However, we know less about (non-integral) rational solutions and even nothing about irrational solutions in $\mathcal{F}^-$. Thus we pose the following.

Problem 11.4 For rational solutions $\lambda = (p, q, r; a, b; x) \in \mathcal{F}^-$ how large can the least common denominator of p, $q \in \mathbb{Q}$ become ? Moreover, as mentioned in item (3) of Remark 3.13 we can also ask: Is there any irrational solution in $\mathcal{F}^-$?

These are among many interesting problems in searching for, or rather confinement of, or perhaps proving the nonexistence of "strange" hypergeometric identities (see also [2] for a related discussion).

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References

[1] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge Univ. Press, Cambridge, 1999.

[2] M. Apagodu and D. Zeilberger, *Searching for strange hypergeometric identities by sheer brute force*, Integers **8** (2008), A36, 6 pages.

[3] A. Ebisu, *Three term relations for the hypergeometric series*, Funkcial. Ekvac. **55** (2012), 255–283.

[4] A. Ebisu, *On a strange evaluation of the hypergeometric series by Gosper*, Ramanujan J. **32** (2013), 101–108.

[5] A. Ebisu, *Special values of the hypergeometric series*, e-Print arXiv: `1308.5588`.

[6] A. Erdélyi, ed., *Higher Transcendental Functions, Vol. I*, McGraw-Hill, New York, 1953.

[7] I. Gessel and D. Stanton, *Strange evaluations of hypergeometric series*, SIAM J. Math. Anal. **13** (1982), no. 2, 295–308.

[8] K. Iwasaki, *On some hypergeometric summations*, e-Print arXiv: `1408.5658`, ver. 3.

[9] W. Koepf, *Algorithms for m-fold hypergeometric summation*, J. Symbolic Comput. **20** (1995), no. 4, 399–417.

[10] E. Kummer, *Über die hypergeometrische Reihe* $1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots$, J. Reine Angew. Math. **15** (1836), 39–83.

[11] R. Maier, *A generalization of Euler's hypergeometric transformation*, Trans. Amer. Math. Soc. **358** (2005), no. 1, 39–57.

[12] R. Vidunas, *Contiguous relations of hypergeometric series*, J. Comput. Appl. Math. **153** (2003), no. 1-2, 507–519.

[13] R. Vidunas, *Dihedral Gauss hypergeometric functions*, Kyushu J. Math. **65** (2011), 141–167.

[14] H.S. Wilf and D. Zeilberger, *Rational functions certify combinatorial identities*, J. Amer. Math. Soc. **3** (1990), 147–158.

[15] J. Wimp, *Irreducible recurrences and representation theorems for* ${}_3F_2(1)$, Comput. Math. Appl. **9** (1983), no. 5, 669–678.

[16] D. Zeilberger, *Gauss's* ${}_2F_1(1)$ *cannot be generalized to* ${}_2F_1(x)$, J. Comput. Appl. Math. **39** (1992), no. 3, 379–382.